General-Purpose Tools in Algebra

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http://www.edc.org/cmeproject

CME PROJECT

The next generation of NSF-funded high school mathematics programs.

For these slides and others

www.edc.org/cmeproject

Summer Workshops
August 4-8, 2008

The Utility of Mathematics

Mathematics constitutes one of the most ancient and noble intellectual traditions of humanity. It is an enabling discipline for all of science and technology, providing powerful tools for analytical thought as well as the concepts and language for precise quantitative description of the world around us. ***It affords knowledge and reasoning of extraordinary subtlety and beauty, even at the most elementary levels.***

RAND Mathematics Study Panel, 2002

CME Project Overview

Fundamental Organizing Principle

The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more importantly, from developing the ways of thinking—the *habits of mind*—used to create the results.

CME Project Overview

"Traditional" course structure: it's familiar but different

- Structured around the sequence of Algebra 1, Geometry, Algebra 2, Precalculus
- Uses a variety of instructional approaches
- Focuses on particular mathematical habits
- Uses examples and contexts from many fields
- Organized around mathematical themes

CME Project Overview

An early meeting…

"I'd never use a curriculum that has worked-out examples in the student text."

Nancy, Teacher Advisory Board

"I'd never use a curriculum that *doesn't* have worked-out examples in the student text."

Chuck, Teacher Advisory Board

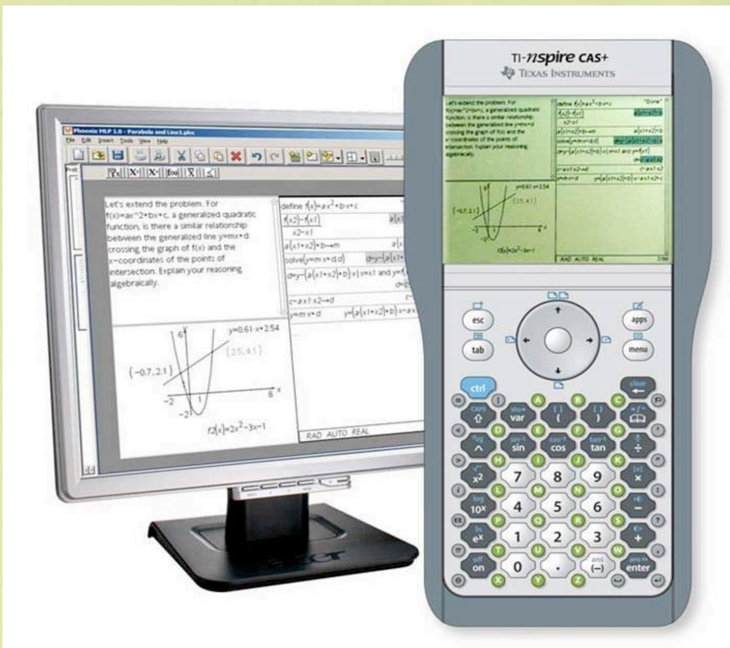
CME Project Overview

CME Project audience: the (large number of) teachers who…

- Want the familiar course structure
- Want a problem- and exploration-based program
- Want to bring activities to “closure”
- Want rigor and accessibility for all

CME Project Overview

Relationship with Texas Instruments



CME Project makes essential use of technology:

- A “function-modeling” language (FML)
- A computer algebra system (CAS)
- An interactive geometry environment

General-Purpose Tools

Develop students' *habits of mind*

- Give students opportunities to learn techniques they can use in later studies
- If a choice of techniques exists, emphasize techniques that have the most general use
- Example: expansion boxes

General-Purpose Tools

Three examples from *CME Project*

- The “guess-check-generalize” method
- Equations as point-testers
- Factoring by chunking

Why do we emphasize these concepts? Where do they lead?

A Word Problem

Nicole drives from Seattle to San Francisco at 60 miles per hour, then from San Francisco to Seattle at 50 miles per hour. The entire trip, both ways, takes a total of 30 hours. How far is it from Seattle to San Francisco?

Hm, this problem would be a lot easier if you gave me the mileage and asked me to calculate the time…

Guess-Check-Generalize

Take a guess at the answer, doesn't have to be a good one… then check it

Let's guess that the distance is 400 miles. The trip to San Francisco takes 6 2/3 hours… the trip to Seattle takes 8 hours. 14 2/3 total. Nope, wrong.

Guess-Check-Generalize

Keep guessing until you can generalize the process

Try 1200 miles…

- Seattle to SF: 1200/60 = 20 hours
- SF to Seattle: 1200/50 = 24 hours
- Total trip: 44 hours, still wrong.

But now I can **write an equation!**

Guess-Check-Generalize

The general "checker" is your equation

If it's *n* miles…

- Seattle to SF: *n*/60 hours
- SF to Seattle: *n*/50 hours
- Total trip must equal 30 hours

$$\frac{n}{60} + \frac{n}{50} = 30$$

Guess-Check-Generalize

Why emphasize this?

Habits of mind…

- Students have a starting point: try numbers
- Students learn more about variables
- Students organize work to see patterns
- Students look for proportional or linear patterns, and make “nice” guesses

This is a **general-purpose tool** instead of a set of special “word problem” tools.

Guess-Check-Generalize

Where does this lead?

- Algebra 1: Graphing; functions
- Geometry: Area-perimeter problems
- Algebra 2: Line of best fit; models for exponential and logarithmic situations
- Precalculus: Models for trigonometric situations; monthly payment on a car loan

An Ellipse

Here's the equation of an ellipse:

$$9x^2 – 36x + 4y – 24y + 36 = 0$$

Determine whether or not the point (3,5.6) is on the graph of the ellipse.

In 12th-grade classrooms, we've seen students squint at their perfectly-drawn graphs to try and decide. These capable students have missed an important concept…

Equations: Point-Testers

The graph of an equation is precisely those points that make the equation true.

The connection between graphs and equations is explicit and repeated as frequently as possible in ***CME Project***.

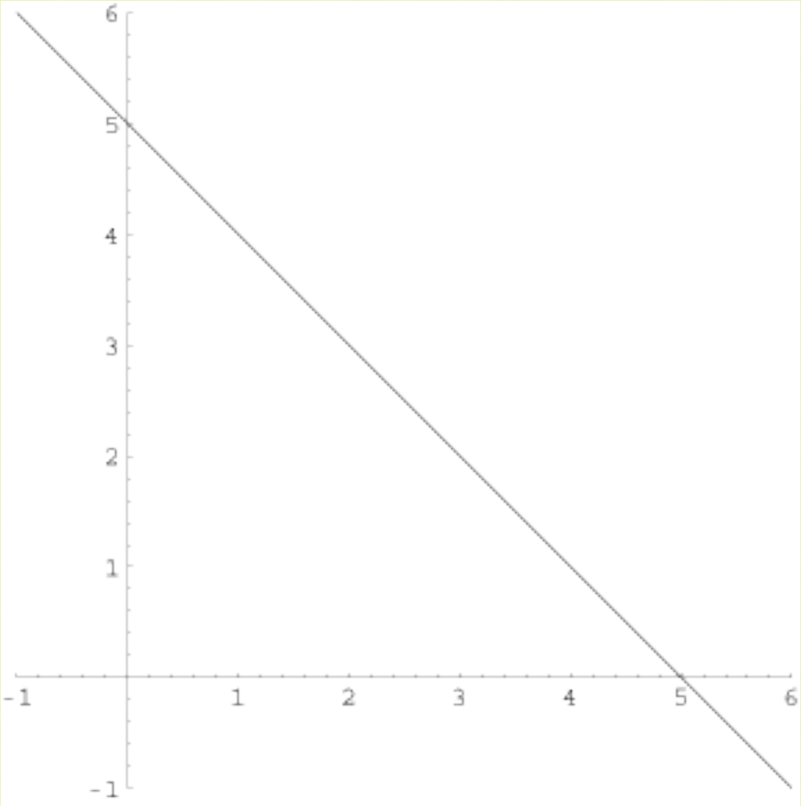
Examples from Algebra 1, Chapter 3…

Equations: Point-Testers

Is (1,4) on the graph of $x + y = 5$?

$1 + 4 = 5$

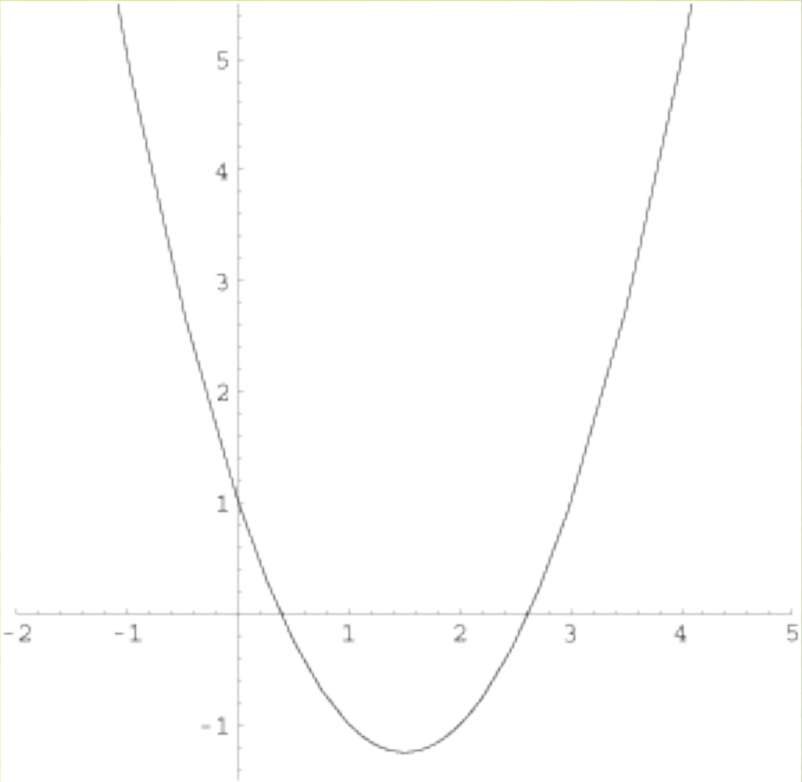
$5 = 5$ ✓



Equations: Point-Testers

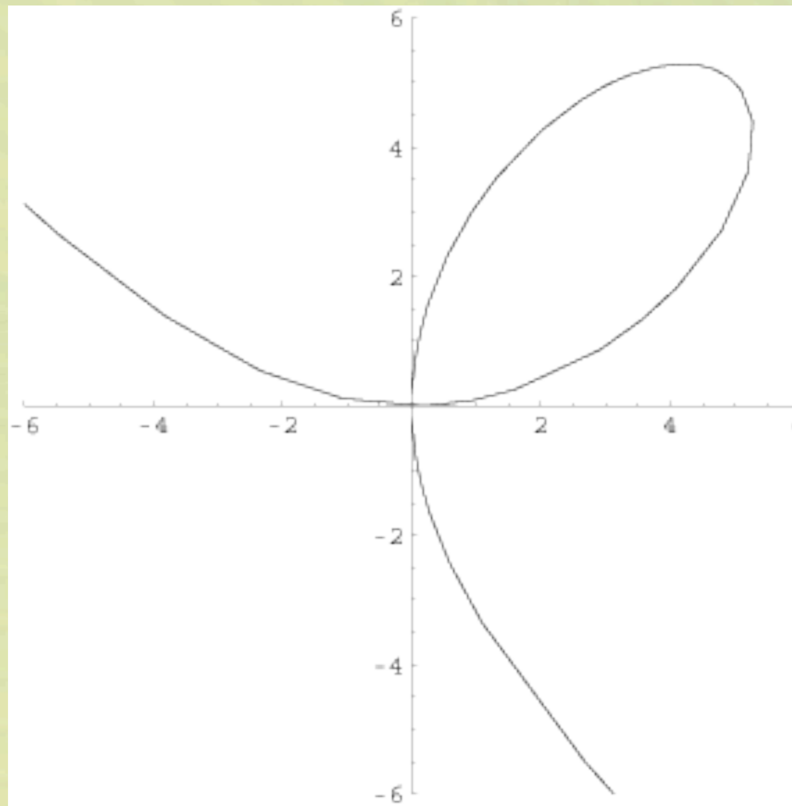
Is (-1,5) on the graph of
$y = x^2 - 3x + 1$?

$5 = (-1)^2 - 3(-1) + 1$
$5 = 1 + 4 + 1$
$5 = 6$ ✗



Equations: Point-Testers

Is (5,5) on the graph of $x^3 + y^3 = 10xy$?



$5^3 + 5^3 = 10(5)(5)$

$125 + 125 = 250$

$250 = 250$ ✓

Equations: Point-Testers

Students learn about the general connection between equations and graphs before learning any specific types of graphs.

Students determine “point-testers” to find equations that match graphs, based on the behavior of the graph.

- Which is vertical, $x = 1$ or $y = 1$?
- What is the graph of $y = x$?

Equations of Lines

CME Project starts with the slope between two points, then assumes that three points are collinear if the slope between any two of them is the same.

Consider the line connecting (4,1) and (5,3). Is (7,6) on this line?

Equations of Lines

Is the point (7,6) on the line through (4,1) and (5,3)?

$$\frac{3-1}{5-4} = \frac{2}{1} = 2 \qquad \frac{6-1}{7-4} = \frac{5}{3}$$

They aren't the same, so (7,6) is not on the line.

Equations of Lines

A point is on the line if and only if its slope to (4,1) is 2.

Students continue to **test points** until they are ready to generalize:

Is the point (x,y)
on the line through
(4,1) and (5,3)?

Equations of Lines

Is the point (x,y) on the line through (4,1) and (5,3)?

$$\frac{3-1}{5-4} = \frac{2}{1} = 2 \qquad \frac{y-1}{x-4} = 2$$

so (x,y) is on the line when $\frac{y-1}{x-4} = 2$

$y - 1 = 2(x - 4)$

Teacher Comment

From our Algebra 1 field test...

“One student suggested that once you find the slope (say, $^1/_2$), you could write $y = {}^1/_2x$, [but] she didn't know what to do with that…. I [reminded] the students [about] equations as point-testers and asked her what we might do from here.

Teacher Comment

She suggested plugging the point in for *x* and *y*. WOW!

I said, OK, but it doesn't satisfy the equation, and it has to, so what might we do? She suggested finding an adjustment amount to make it work. BINGO!

Teacher Comment

I got so excited, the students were very concerned! I've never had a *student* come up with how to use slope-intercept form to find the equation of a line before—all I've gotten [were] blank stares!"

- *Annette Roskam*
  Rice Lake High School
  Rice Lake, Wisconsin

Equations: Point-Testers

Why emphasize this?

Habits of mind…

- Students learn the key relationship between an equation and its graph ASAP
- Students learn that the equation of a line comes from how the line is characterized

Equations: Point-Testers

Why emphasize this?

Habits of mind…

- Students learn the utility of knowing more than one available form for an equation

This is a **general-purpose tool** that students can apply to other situations.

Equations: Point-Testers

Where does this lead?

- Algebra 1: Intersections, inequalities, function graphs, the equation of a quadratic
- Geometry: Coordinate geometry
- Algebra 2: Factor Theorem, exponential and trigonometric functions, transformations
- Precalculus: Locus definition of conics, tangent lines to functions

Factoring: Quadratics

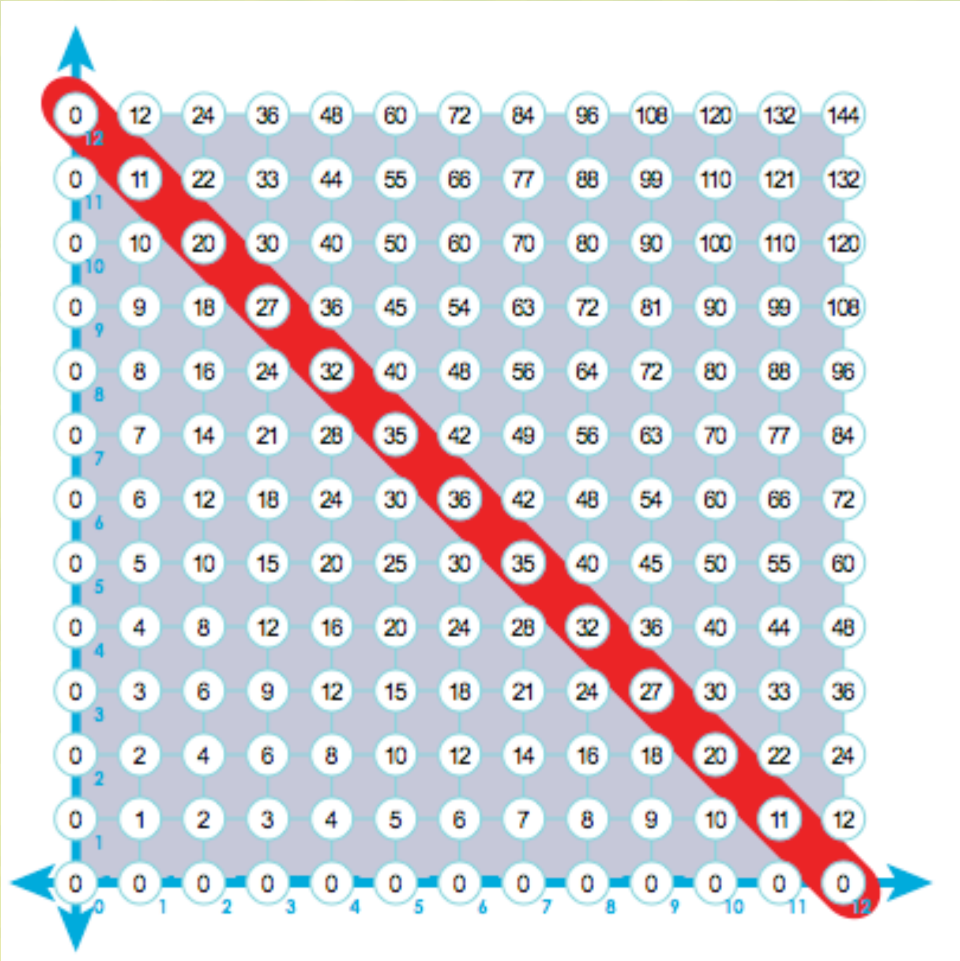
Monic quadratics:

"Sum-Product" problems

$$x^2 + 14x + 48$$

Find two numbers whose sum is 14 and product is 48.

Factoring: Quadratics



Factoring: Quadratics

What about this one?

$$49x^2 + 35x + 6$$

$$(7x)^2 + 5(7x) + 6$$

$$(\text{✍})^2 + 5(\text{✍}) + 6$$

$$z^2 + 5z + 6$$

$$(z+3)(z+2)$$

$$(7x+3)(7x+2)$$

Factoring: Quadratics

What about this one?

$$6x^2 + 31x + 35$$

$$6\left(6x^2 + 31x + 35\right)$$

$$\left(6x\right)^2 + 31\left(6x\right) + 210$$

$$z^2 + 31z + 210$$

Factoring: Quadratics

$$z^2 + 31z + 210$$

$$(z + 21)(z + 10)$$

$$(6x + 21)(6x + 10)$$

$$3(3x + 7) \cdot 2(2x + 5)$$

$$\cancel{6}(3x + 7)(2x + 5)$$

Factoring: Quadratics

Why emphasize this?

Habits of mind…

- Students learn to “chunk” expressions
- Students learn to reduce problems to simpler ones
- Students learn to look for structure in algebraic expressions

This is a **general-purpose tool** that is useful throughout mathematics.

Factoring: Quadratics

Where does this lead?

- Algebra 1: Solving quadratic equations, deriving the quadratic formula
- Geometry: Pythagorean Theorem, area problems
- Algebra 2: Advanced factoring, solving polynomial equations, Heron's formula
- Precalculus: Trigonometric equations, finding closed forms for recurrences

An Input-Output Table

Find *more than one* way to define a function that matches the table at right.

Input n	Output $f(n)$
0	3
1	8
2	13
3	18
4	23

Using Differences

n	$f(n)$	Δ
0	3	5
1	8	5
2	13	5
3	18	5
4	23	

Two rules emerge:

$f(n) = 5n + 3$

$f(n) = f(n-1) + 5,\ f(0) = 3$

Which is "right"?

Which is more useful?

(Both.)

Function Modeling

Use the *n*Spire to define this function… in either form!

To define $f(n) = 5n + 3$:

1. Hit the HOME icon (upper right), then choose option 1, Calculator.
2. Push MENU, then option 1, choice 1, Define.
3. Type "$f(n) = 5n + 3$" (without quotes) and hit ENTER. Response: *Done.*
4. Test: type $f(5)$ and hit ENTER.

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Function Modeling

To define $g(n) = \begin{cases} 3, & n = 0 \\ g(n-1) + 5, & n > 0 \end{cases}$

1. Push MENU, then option 1, choice 1, Define.
2. Type "*g*(*n*) =" (without quotes). Do *not* hit ENTER yet.
3. Push the blue CTRL button, then the multiplication symbol on the right, to bring up a template screen. (Cool!)
4. Select the piecewise function template, seventh from left on the top row.

Function Modeling

To define $g(n) = \begin{cases} 3, & n = 0 \\ g(n-1) + 5, & n > 0 \end{cases}$

5. Type in the first box: 3, then hit the TAB button (upper left) to move.
6. Type in the second box: n=0, then TaB.
7. Type in the third box: g(n-1) + 5.
8. Type in the fourth box: n>0.
9. Hit ENTER to define the function. Response: *Done*.
10. Test: type g(5) and hit ENTER.

Function Modeling

Are *f* and *g* "equal"?

- Issues of domain arise naturally: domain of *f* and *g* are different
- Issues of technology arise: *g* stops working after a while, but whose fault?

This question revisited, finally resolved in Precalculus with use of induction

Functions as Objects

Students begin to use functions as primitives

- *f*(*x*+3) is a function; so is 3*f*(*x*)
- Lagrange Interpolation (Algebra 2, Chapter 2) relies on function primitives
- Focus on algebraic structure of functions
  - identity? inverses? commutativity? linearity?

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Function Modeling

Why emphasize this?

Habits of mind…

- Deep, continued exploration of tables and functions revisited several times
- Students work with functions like numbers
- Students build intuition about functions before formalization in Algebra 2
- Relationships between closed-form and recursive functions are seen frequently

Function Modeling

Where does this lead?

- Algebra 1: Intuitive definitions; simple identities using factoring
- Geometry: Approximating *pi*
- Algebra 2: Functions that match tables; Lagrange Interpolation; summations
- Precalculus: Identities; induction; Newton’s Difference Formula

Teacher Comment

From our Algebra 2 field test…

"I didn't really want to bother [with Lagrange Interpolation]… but I couldn't believe the connections my students made when they started working on it. I was floored – they made connections that they had never made before. They understood how to add functions, and why you might want to. They understood that functions are things you *can* add. And what surprised me most of all was how much they loved solving the problems – because they were good at it."

Chris Martino, Arlington High School

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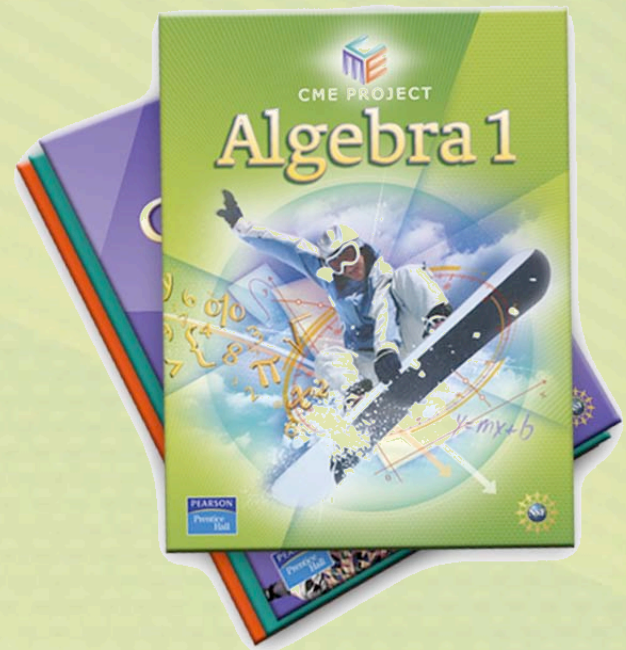
CME Project Availability Dates

- Algebra 1, Geometry, and Algebra 2
  - Available right now!
- Precalculus
  - Available Summer 2008



CME Project Workshops

Developing Habits of Mind Workshops

- August 4-8, 2008 in Boston
- Explore mathematics content using *CME Project* materials
- Learn about pedagogical tools and style including mathematical representations, word problems, and skills practice
- Address issues of implementation, differentiation, and assessment
- Network with educators from across the country

www.edc.org/cmeproject

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CME Project

For more information

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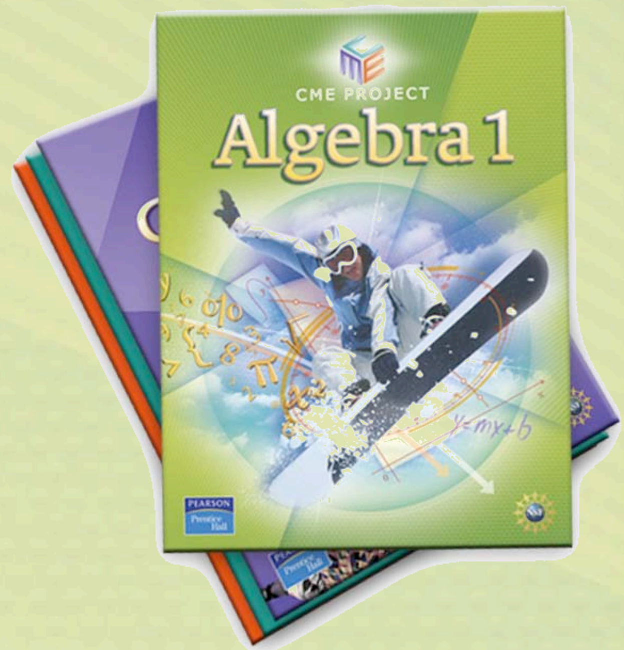

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