Beyond Topics

Some Organizing Principles for a Coherent Approach to Algebra

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Slides available at
`www.edc.org/cmeproject`

Outline

THE NATIONAL MATHEMATICS PANEL

The Panel recommends that school algebra be consistently understood in terms of the Major Topics of School Algebra given in Table 1.

The National Mathematics Panel

Table 1

Symbols and Expressions

- Polynomial expressions
- Rational expressions
- Arithmetic and finite geometric series

Linear Equations

- Real numbers as points on the number line
- Linear equations and their graphs
- Solving problems with linear equations
- Linear inequalities and their graphs
- Graphing and solving systems of simultaneous linear equations

The National Mathematics Panel
Table 1

Quadratic Equations

- Factors and factoring of quadratic polynomials with integer coefficients
- Completing the square in quadratic expressions
- Quadratic formula and factoring of general quadratic polynomials
- Using the quadratic formula to solve equations

Functions

- Linear functions
- Quadratic functions—word problems involving quadratic functions
- Graphs of quadratic functions and completing the square

NATIONAL MATHEMATICS PANEL
TABLE 1

FUNCTIONS CONT'D

- Polynomial functions (including graphs of basic functions)
- Simple nonlinear functions (e.g., square and cube root functions; absolute value; rational functions; step functions)
- Rational exponents, radical expressions, and exponential functions
- Logarithmic functions
- Trigonometric functions
- Fitting simple mathematical models to data

National Mathematics Panel

Table 1

Algebra of Polynomials

- Roots and factorization of polynomials
- Complex numbers and operations
- Fundamental theorem of algebra
- Binomial coefficients (and Pascal's Triangle)
- Mathematical induction and the binomial theorem

Combinatorics and Finite Probability

- Combinations and permutations, as applications of the binomial theorem and Pascal's Triangle

National Mathematics Panel

A detailed mathematical discussion of many of these topics is given in the accompanying task force report.

http://www.ed.gov

What can happen

"Factoring Pattern for $x^2 + bx + c$, c Negative"

Factor. Check by multiplying factors. If the polynomial is not factorable, write "prime."

1. $a^2 + 4a - 5$	**2.** $x^2 - 2x - 3$	**3.** $y^2 - 5y - 6$
4. $b^2 + 2b - 15$	**5.** $c^2 - 11c - 10$	**6.** $r^2 - 16r - 28$
7. $x^2 - 6x - 18$	**8.** $y^2 - 10c - 24$	**9.** $a^2 + 2a - 35$
10. $k^2 - 2k - 20$	**11.** $z^2 + 5z - 36$	**12.** $r^2 - 3r - 40$
13. $p^2 - 4p - 21$	**14.** $a^2 + 3a - 54$	**15.** $y^2 - 5y - 30$
16. $z^2 - z - 72$	**17.** $a^2 - ab - 30b^2$	**18.** $k^2 - 11kd - 60d^2$
19. $p^2 - 5pq - 50q^2$	**20.** $a^2 - 4ab - 77b^2$	**21.** $y^2 - 2yz - 3z^2$
22. $s^2 + 14st - 72t^2$	**23.** $x^2 - 9xy - 22y^2$	**24.** $p^2 - pq - 72q^2$

STATE FRAMEWORKS

"Describe, complete, extend, analyze, generalize, and create a wide variety of patterns, including iterative, recursive (e.g., Fibonacci Numbers), linear, quadratic, and exponential functional relationships."

Whew. . .

WHAT CAN HAPPEN

n	1	2	3	4	5	6
t_n	3	5				

The first two terms of a sequence, t_1 and t_2 , are shown above as 3 and 5. Using the rule: $t_n = t_{n-1} + t_{n-2}$, where $n \geq 3$, complete the table.

ACHIEVE (ADP) BENCHMARKS

Content is arranged in five strands:

1. Number
2. Discrete Mathematics
3. Algebra
4. Geometry
5. Probability and Statistics

ACHIEVE (ADP) BENCHMARKS

The algebra strand is subdivided into six "clusters:"

(1) Perform basic operations on algebraic expressions fluently and accurately.

(2) Understand functions, their representations, and their properties.

(3) Apply basic algebraic operations to solve equations and inequalities.

(4) Graph a variety of equations and inequalities in two variables, demonstrate understanding of the relationships between the algebraic properties of an equation and the geometric properties of its graph, and interpret a graph.

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Achieve (ADP) Benchmarks

(5) Solve problems by converting the verbal information given into an appropriate mathematical model involving equations or systems of equations, apply appropriate mathematical techniques to analyze these mathematical models, and interpret the solution obtained in written form using appropriate units of measurement.

(6) Understand the binomial theorem and its connections to combinatorics, Pascal's Triangle, and probability.

`http://www.achieve.org`

Focus in High School Mathematics

(FROM THE DRAFT)

A number of documents have been produced over the past few years providing detailed analyses of the topics that should be addressed in each course of high school mathematics. . . .

[FHSM] takes a somewhat different approach, proposing curricular emphases and instructional approaches that make reasoning and sense making foundational to the content that is taught and learned.

Along with the more detailed content recommendations outlined in *Principles and Standards*, [FHSM] provides a critical filter in examining any curriculum arrangement to ensure that the ultimate goals of the high school mathematics program are achieved.

Focus in High School Mathematics: Algebra
(FROM THE DRAFT)

Key elements of reasoning and sense making with algebraic symbols include

- **Meaningful use of symbols**. Choosing variables and constructing expressions and equations in context; interpreting the form of expressions and equations; manipulating expressions so that interesting interpretations can be made.

- **Mindful manipulation**. Connecting manipulation with the laws of arithmetic; anticipating the results of manipulations; choosing procedures purposefully in context; picturing calculations mentally.

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Focus in High School Mathematics

(FROM THE DRAFT)

- **Reasoned solving.** Seeing solution steps as logical deductions about equality; interpreting solutions in context.
- **Connecting algebra with geometry.** Representing geometric situations algebraically and algebraic situations geometrically; using connections in solving problems.
- **Linking expressions and functions.** Using multiple algebraic representations to understand functions; working with function notation.

Focus in High School Mathematics

(FROM THE DRAFT)

A high school curriculum that focuses on reasoning and sense making will help to satisfy the increasing demand for scientists, engineers, and mathematicians while preparing students for whatever professional, vocational, or technical needs may arise.

(More details are in the brochure that came with registration materials)

The Habits of Mind Approach

What mathematicians most wanted and needed from me was ***to learn my ways of thinking****, and not in fact to learn my proof of the geometrization conjecture for Haken manifolds.*

— William Thurston

On Proof and Progress in Mathematics

The Habits of Mind Approach

Mathematics constitutes one of the most ancient and noble intellectual traditions of humanity. It is an enabling discipline for all of science and technology, providing powerful tools for analytical thought as well as the concepts and language for precise quantitative description of the world around us.
It affords knowledge and reasoning of extraordinary subtlety and beauty, even at the most elementary levels.

— Rand Mathematics Study Panel, 2002

CME Fundamental Organizing Principle

The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more importantly, from developing the ways of thinking—the ***habits of mind****—used to create the results.*

Some Goals for School Algebra

- Provide all students with a collection of general purpose algebraic habits
- Prepare students who intend to go into STEM fields with the necessary technical tools
- Give students a sense of algebra as a modern, vibrant, and coherent mathematical discipline that
  - is built on a small number of central ideas
  - has profound utility throughout mathematics, science, and technology
  - provides immense intellectual satisfaction
  - demonstrates the power and applicability of abstraction

General Mathematical Habits

- Performing thought experiments
- Finding and explaining patterns
- Creating and using representations
- Generalizing from examples
- Expecting mathematics to make sense

Analytic/Geometric Habits of Mind

- Reasoning by continuity
- Seeking geometric invariants
- Looking at extreme cases
- Passing to the limit
- Modeling geometric phenomena with continuous functions

Algebraic Habits of Mind

- Seeking regularity in repeated calculations
- “Delayed evaluation”—seeking form in calculations
- “Chunking”—changing variables in order to hide complexity
- Reasoning about and picturing calculations and operations
- Extending operations to preserve rules for calculating
- Purposefully transforming and interpreting expressions
- Seeking and specifying structural similarities

Algebraic Habits of Mind

Claim: Organizing algebra around habits like these can bring coherence to the subject.

Some Evidence: Let's look at some seemingly different topics and see if there are similarities in the mathematical thinking that might help students master them.

Some Thorny Topics in Elementary Algebra

1. Students have trouble expressing generality with algebraic notation.
2. This is especially prevalent when they have to set up equations to solve word problems.
3. Many students have difficulty with slope, graphing lines, and finding equations of lines.
4. Building and using algebraic functions is another place where students struggle.

This list looks like a collection of disparate topics—using notation, solving word problems, ...

Some Thorny Topics in Elementary Algebra

But if one looks underneath the topics to the mathematical habits that would help students master them, one finds a remarkable similarity:

A key ingredient in such a mastery is the reasoning habit of seeking and expressing regularity in repeated calculations.

- This habit manifests itself when one is performing the same calculation over and over and begins to notice the “rhythm” in the operations.
- Articulating this regularity leads to a generic algorithm, typically expressed with algebraic symbolism, that can be applied to any instance and that can be transformed to reveal additional meaning, often leading to a solution of the problem at hand.

Example 1: The Dreaded Algebra Word Problem

Think about how hard it is for students to set up an equation that can be used to solve an algebra word problem. Some reasons for the difficulties include reading levels and unfamiliar contexts. But there has to be more to it than these surface features.

Consider, for example, the following two problems.

Example 1: The Dreaded Algebra Word Problem

1. The driving distance from Boston to Chicago is is 990 miles. Rico drives from Boston to Chicago at an average speed of 50mph and returns at an average speed of 60mph. For how many hours is Rico on the road?

2. Rico drives from Boston to Chicago at an average speed of 50mph and returns at an average speed of 60mph. Rico is on the road for 36 hours. What is the driving distance from Boston to Chicago?

The problems have identical reading levels, and the context is the same in each. But teachers report that many students who can solve problem 1 are baffled by problem 2.

Example 1: The Dreaded Word Problem

This is where the reasoning habit of "expressing the rhythm" in a calculation can be of great use. The basic idea:

- Guess at an answer to problem 2, and
- check your guess as if you were working on problem 1, *keeping track of your steps*.

The purpose of the guess is not to stumble on (or to approximate) the correct answer; rather, it is to help you construct a "checking algorithm" that will work for any guess.

Example 1: The Dreaded Word Problem

Problem 2: He drives over at an average speed of 50mph and returns at an average of 60mph. He's on the road for 36 hours. What is the driving distance?

So, you take several guesses until you are able to express your checking algorithm in algebraic symbols. For example:

- Suppose the distance is 1000 miles.
- How do I check the guess of 1000 miles? I divide 1000 by 50. Then I divide 1000 by 60. Then I add my answers together, to see if I get 36. I don't.
- So, check another number—say 950. 950 divided by 50 plus 950 divided by 60. Is that 36?
- No, but a general method is evolving that will allow me to check *any* guess.

Example 1: The Dreaded Word Problem

- My guess-checker is

$$\frac{\text{guess}}{50} + \frac{\text{guess}}{60} \stackrel{?}{=} 36$$

- So my *equation* is

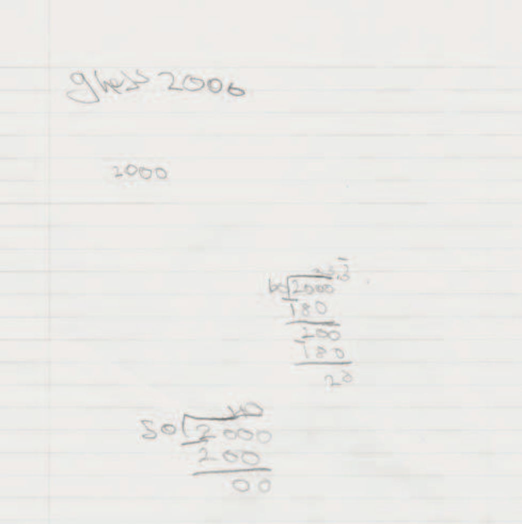
$$\frac{\text{guess}}{50} + \frac{\text{guess}}{60} = 36$$

or, letting x stand for the unknown correct guess,

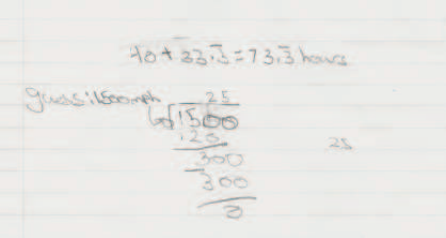
$$\frac{x}{50} + \frac{x}{60} = 36$$

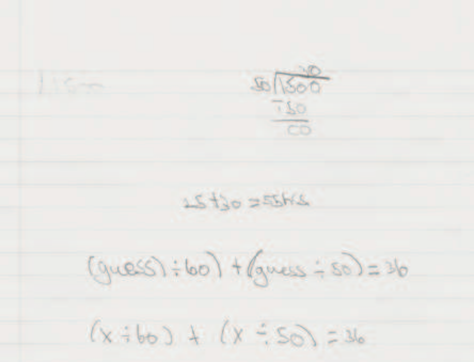
Example 1: The Dreaded Word Problem

Here's some student work that shows how the process develops:



Example 1: The Dreaded Word Problem





Example 2: Equations for lines

Why is “linearity” so hard for students?

- Results from a research project undertaken as part of the *Focus on Mathematics* partnership in Boston suggests several reasons.
- One is that many students think that $y = 3x + 4$ is a “code” for “put a point at $(0, 4)$, then go over 1 and up 3, put a point there, and then draw line between these two points.”
- Missing is the idea that a point is on the graph of an equation if and only if its coordinates satisfy the equation.
- The habit of abstracting from computations can help here, too.

Example 2: Equations for lines

- Suppose a student, new to algebra and with no formulas in tow, is asked to find the equation of the vertical line ℓ that passes through $(5, 4)$.
- Students can draw the line, and, just as in the word problem example, they can guess at some points and check to see if they are on ℓ.
- Trying some points like $(5, 1)$, $(3, 4)$, $(2, 2)$, and $(5, 17)$ leads to a generic guess-checker:

  To see if a point is on ℓ, you check that its x-coordinate is 5.

- This leads to a guess-checker: $x \stackrel{?}{=} 5$ and the equation

$$x = 5$$

Example 2: Equations for lines

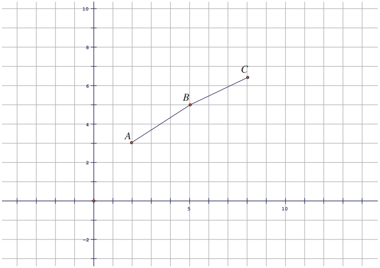
- What about lines for which there is no simple guess-checker? The idea is to find a geometric characterization of such a line and then to develop a guess-checker based on that characterization. One such characterization uses *slope*.

- In first-year algebra, students study slope, and one fact about slope that often comes up is that three points on the coordinate plane, not all on the same vertical line, are collinear if and only if the slope between any two of them is the same.

Example 2: Equations for lines

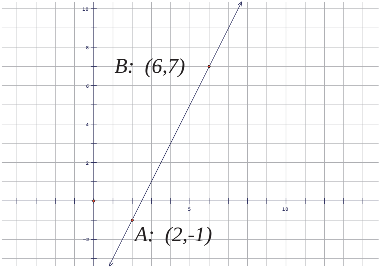
If we let $m(A, B)$ denote the slope between A and B (calculated as change in y-height divided by change in x-run), then the collinearity condition can be stated like this:

Basic assumption: A, B, and C are collinear $\Leftrightarrow m(A, B) = m(B, C)$



Example 2: Equations for lines

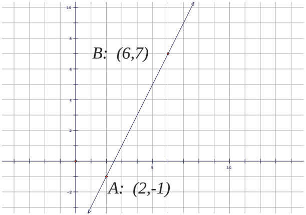
What is an equation for $\ell = \overleftrightarrow{AB}$ if $A = (2, -1)$ and $B = (6, 7)$?



Try some points, keeping track of the steps...

Example: Equations for lines

- $A = (2, -1)$ and $B = (6, 7)$
- $m(A, B) = 2$



- Test $C = (3, 4)$:
  $m(C, B) = \frac{4-7}{3-6} \stackrel{?}{=} 2 \Rightarrow$ Nope
- Test $C = (5, 5)$:
  $m(C, B) = \frac{5-7}{5-6} \stackrel{?}{=} 2 \Rightarrow$ Yup
- The “guess-checker?”
  Test $C = (x, y)$:
  $m(C, B) = \frac{y-7}{x-6} \stackrel{?}{=} 2$

And an equation is $\frac{y-7}{x-6} = 2$

Other Examples Where This Habit Is Useful

- Finding lines of best fit
- Fitting functions to tables of data
- Deriving the quadratic formula
- Establishing identities in Pascal's triangle
- Using recursive definitions in a CAS or spreadsheet

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FROM THE DRAFT OF *FHSM*

Although a long-term goal of algebraic learning is a fluid, nearly automatic facility with manipulating algebraic expressions. . . this ease can be best achieved by first learning to pay close attention to interpreting expressions. . . .

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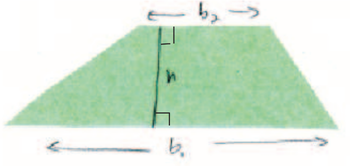
As comfort with expressions grows, constructing and interpreting them requires less and less effort and gradually become almost subconscious. The true foundation for algebraic manipulation is close attention to meaning and structure.

Example 1: Area Formulas

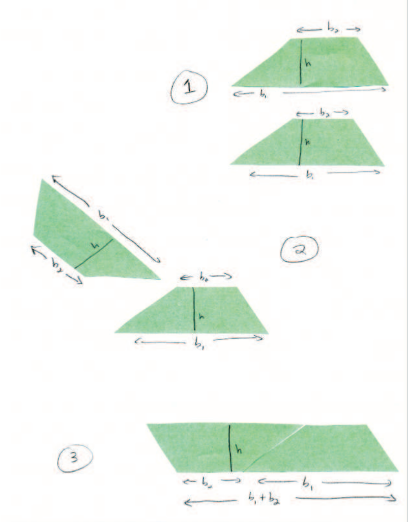
Imagine a class in which students know the formulas for the areas of rectangles and parallelograms. How could the class use this to reason out the formula for the area, A, of a trapezoid?



One way is to turn the trapezoid into a parallelogram, keeping track of changes in the linear dimensions. Here are three methods that have come up in various classrooms:

Example 1: Area Formulas

Duplicate the trapezoid and arrange the pieces to make a parallelogram:



Students who use this argument might reason that, because the area of the parallelogram is $2A$,

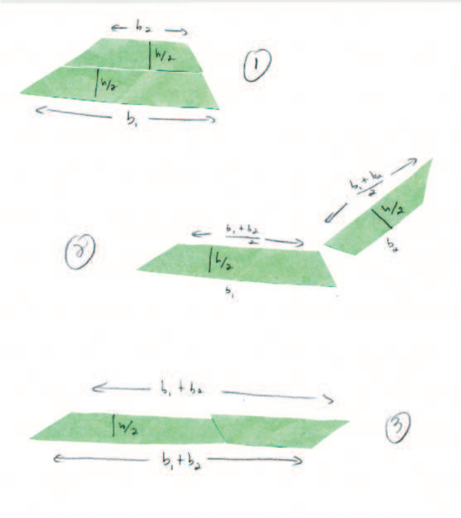
$$2A = (b_1 + b_2)h$$

and

$$A = \frac{1}{2}\left[(b_1 + b_2)h\right]$$

EXAMPLE 1: AREA FORMULAS

Slice the trapezoid along its midline and arrange the pieces to make a parallelogram:

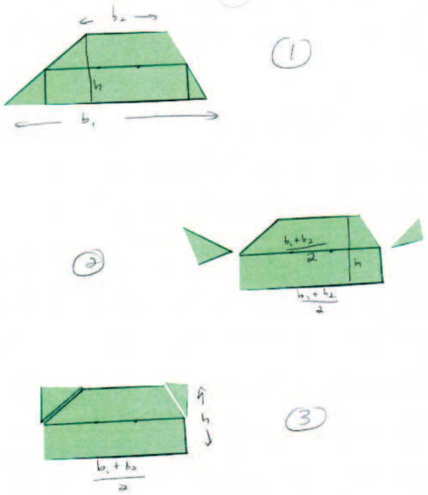


The resulting parallelogram's height is half the height of the trapezoid, and the length of its base is the sum of the lengths of the trapezoid's bases. So, a student who uses this line of reasoning might express the area in this way:

$$A = (b_1 + b_2)\left(\frac{h}{2}\right)$$

Example 1: Area Formulas

Slice the trapezoid along its midline and then cut off two triangles, folding up the pieces to make a rectangle:



The trapezoid and rectangle have the same area and the same height, but the rectangle's base is congruent to the *midline* of the trapezoid, so its length is half the sum of the bases of the trapezoid. Students are led to

$$A = \left(\frac{b_1 + b_2}{2}\right) h$$

Example 1: Area Formulas

So, a class could come up with at least three expressions for the same area A:

$$\frac{1}{2}\left[(b_1+b_2)h\right], \quad (b_1+b_2)\left(\frac{h}{2}\right), \text{ and } \quad \left(\frac{b_1+b_2}{2}\right)h$$

Algebraically, these three expressions are all equivalent—many algebra students can get from one to the other using the basic rules for calculating with algebraic expressions. This activity allows students to hang this equivalence on a concrete geometric hook:

Example 1: Area Formulas

- Students don't *start* with the task of showing that the expressions are algebraically equivalent.
- Rather, different algorithms for calculating the same thing emerge from different lines of reasoning in the class—in this example, the same area can be calculated in several different ways.
- Algebra is the perfect tool for showing that the different-looking expressions will *always* produce the same result when the same values are substituted for the variables in each.

Example 1: Area Formulas

- One can turn the tables and ask students to come up with geometric interpretations of expressions that are algebraically equivalent but that describe different calculations. They might take different forms for the the formula for the area of a triangle, say:

$$\frac{1}{2}bh, \quad \left(\frac{b}{2}\right)h, \quad \text{and} \quad b\left(\frac{h}{2}\right)$$

- Students can pick one and show how a triangle can be moved, copied, or dissected to get a parallelogram whose area calculation mimics the chosen expression.

Example 2: Heron's formula

Heron's Formula is a formula for the area of a triangle in terms of the lengths of its three sides:

If the sides of a triangle have lengths a, b, and c, the area A of the triangle is

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$

Heron's formula can also be written in a symmetric form

$$A = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

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Example 2: Heron's formula

$$A = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

Some questions that ask students to interpret the symbols:

- Use Heron's formula to find the area of an equilateral triangle whose side has length 10. Check this against the result of the formula $A = \frac{1}{2}bh$.
- Under what conditions (on a, b, and c) will Heron's formula produce 0? What kinds of triangles do you get for these inputs?

Example 2: Heron's formula

- We could think of Heron's formula as a function of three inputs:

$$f(a,b,c) = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

Show that

$$\begin{aligned} f(a,b,c) &= f(a,c,b) = f(b,a,c) \\ &= f(b,c,a) = f(c,a,b) = f(c,b,a) \end{aligned}$$

Why does this make sense geometrically?

Example 2: Heron's formula

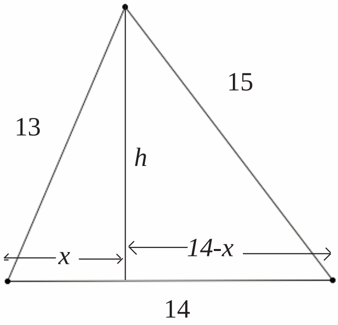
- What is the value of

$$\frac{f(5a,5b,5c)}{f(a,b,c)}$$

  Why does this make sense geometrically?

- Students with more technical algebraic skill can *derive* Heron's formula.

Example 2: Heron's Formula



A good ramp-up to an algebraic proof is to carry out the calculation with numbers, delaying the evaluation so that you can get the form of the result and concentrate on the rhythm of the calculations.

Other Examples Where This Habit Is Useful

- A square maximizes area for a fixed perimeter:

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$$

- The arithmetic-geometric mean inequality:

$$\sqrt{ab} \leq \frac{a+b}{2} \qquad \text{and} \qquad \sqrt[3]{abc} \leq \frac{a+b+c}{3}$$

- The factor and remainder theorems from algebra 2.
- The machine formula for variance:

$$\sigma^2 = \overline{x^2} - (\overline{x})^2$$

$$\vdots$$

Example 1: Factoring in Algebra 1

Factoring monic quadratics:

"Sum-Product" problems

$$x^2 + 14x + 48$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

so...

Find two numbers whose sum is 14 and whose product is 48.

$$(x + 6)(x + 8)$$

Example 1: Factoring in Algebra 1

What about this one?

$$49x^2 + 35x + 6$$

$$\begin{aligned} 49x^2 + 35x + 6 &= (7x)^2 + 5(7x) + 6 \\ &= \clubsuit^2 + 5\clubsuit + 6 \\ &= (\clubsuit + 3)(\clubsuit + 2) \\ &= (7x + 3)(7x + 2) \end{aligned}$$

Example 1: Factoring in Algebra 1

What about this one?

$$6x^2 + 31x + 35$$

$$\begin{aligned}
6(6x^2 + 31x + 35) &= (6x)^2 + 31(6x) + 210 \\
&= \clubsuit^2 + 31\clubsuit + 210 \\
&= (\clubsuit + 21)(\clubsuit + 10) \\
&= (6x + 21)(6x + 10) \\
&= 3(2x + 7) \cdot 2(3x + 5) \\
&= 6(2x + 7)(3x + 5) \quad \textit{so} \ldots
\end{aligned}$$

$$6(6x^2 + 31x + 35) = 6(2x + 7)(3x + 5)$$

EXAMPLE 1: FACTORING IN ALGEBRA 1

What about this one?

$$6x^2 + 31x + 35$$

$$\begin{aligned} 6(6x^2 + 31x + 35) &= (6x)^2 + 31(6x) + 210 \\ &= \clubsuit^2 + 31\clubsuit + 210 \\ &= (\clubsuit + 21)(\clubsuit + 10) \\ &= (6x + 21)(6x + 10) \\ &= 3(2x + 7) \cdot 2(3x + 5) \\ &= 6(2x + 7)(3x + 5) \quad \textit{so} \ldots \end{aligned}$$

$$\cancel{6}(6x^2 + 31x + 35) = \cancel{6}(2x + 7)(3x + 5)$$

EXAMPLE 1: FACTORING IN ALGEBRA 1

What about this one?

$$6x^2 + 31x + 35$$

$$\begin{aligned} 6(6x^2 + 31x + 35) &= (6x)^2 + 31(6x) + 210 \\ &= \clubsuit^2 + 31\clubsuit + 210 \\ &= (\clubsuit + 21)(\clubsuit + 10) \\ &= (6x + 21)(6x + 10) \\ &= 3(2x + 7) \cdot 2(3x + 5) \\ &= 6(2x + 7)(3x + 5) \quad \textit{so} \ldots \end{aligned}$$

$$6x^2 + 31x + 35 = (2x + 7)(3x + 5)$$

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Example 2: Factoring Across the Grades

The *CMP* Factor Game

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30

Example 2: Factoring Across the Grades

A High School Version

$x-1$	x^2-1	x^3-1	x^4-1	x^5-1
x^6-1	x^7-1	x^8-1	x^9-1	$x^{10}-1$
$x^{11}-1$	$x^{12}-1$	$x^{13}-1$	$x^{14}-1$	$x^{15}-1$
$x^{16}-1$	$x^{17}-1$	$x^{18}-1$	$x^{19}-1$	$x^{20}-1$
$x^{21}-1$	$x^{22}-1$	$x^{23}-1$	$x^{24}-1$	$x^{25}-1$
$x^{26}-1$	$x^{27}-1$	$x^{28}-1$	$x^{29}-1$	$x^{30}-1$

Scratchpad

Example 2: Factoring Across the Grades

Things that have come up in class;

- "It's the same as the middle school factor game."
- if m is a factor of n, $x^m - 1$ is a factor of $x^n - 1$

$$\begin{aligned}
x^{12} - 1 &= \left(x^3\right)^4 - 1 \\
&= (\clubsuit)^4 - 1 \\
&= \left(\clubsuit^2 - 1\right)\left(\clubsuit^2 + 1\right) \\
&= (\clubsuit - 1)(\clubsuit + 1)\left(\clubsuit^2 + 1\right) \\
&= \left(x^3 - 1\right)\left((x^3) + 1\right)\left((x^3)^2 + 1\right) \\
&= \left(x^3 - 1\right)(\text{stuff})
\end{aligned}$$

Example 2: Factoring Across the Grades

- If $x^m - 1$ is a factor of $x^n - 1$, m is a factor of n

  This is much harder. One approach uses De Moivre's theorem and with *roots of unity*: complex numbers that are the roots of the equation

$$x^n - 1 = 0$$

Other Examples Where This Habit Is Useful

- Completing the square and removing terms
- Solving trig equations
- Analyzing conics and other curves
- All over calculus
- Interpreting results from a CAS

⋮

CONCLUSIONS

Some conclusions:

- Organizing precollege algebra solely around lists of topics and low-level skills hides the essential coherence of the subject.
- Such organizations lend themselves to implementations that don't convey the spirit of the discipline.
- An organization around algebraic habits of mind has the potential to
  - create programs that are faithful to algebra as a scientific discipline
  - help students experience the power and utility of abstraction
  - provide students with skills useful in many post-secondary endeavors
  - support many different curricular and topical organizations

Thanks

Beyond Topics

Some Organizing Principles for a Coherent Approach to Algebra

Al Cuoco
Center for Mathematics Education, EDC

NCTM 2009 Annual Meeting

Slides available at
`www.edc.org/cmeproject`