Some Tested Approaches to Topics in High School Mathematics

NCTM 2006

Anna Baccaglini-Frank, Al Cuoco, Bowen Kerins,
Sarah Sword, Audrey Ting, Kevin Waterman

Center for Mathematics Education

Education Development Center

Ideas from *The CME Project*

a comprehensive four-year NSF-funded
high school curriculum
forthcoming from Prentice Hall
with additional support from
Texas Instruments

http:/www.edc.org/cmeproject

The CME Project approach

- Traditional course structure: it’s familiar but different
- Each course focuses on particular mathematical habits
- Each course uses examples and contexts from many fields
- Each course is organized around mathematical themes

The Audience

The large number of teachers who:

- Want a problem- and exploration-based program
- Want to bring activities to “closure”
- Want the familiar course structure

Features of *The CME Project*

- It is faithful to mathematics as a discipline
- It highlights the uses of mathematics
- It puts emphasis on mathematical thinking
- It pays attention to student thinking
- Students use the text as both a guide and as a reference

Features of *The CME Project*

- It provides rigorous mathematics for all
- It provides a coherent program
- Statistics and data are integrated throughout
- It makes essential use of technology:
  - A functional modeling language
  - A dynamic geometry environment
  - A Computer Algebra System
- It slices, dices, and does the ironing

Three Themes

EXTENSION:

- Extending number systems
- Extending the domain of functions or operations
- Extending definitions

Three Themes

ENCAPSULATION

- Finding equations for geometric objects
- Building functions to model situations
- Refining calculations into algorithms

Three Themes

PROOF

- Proof as generalized instance
- Proof in geometry
- Proof by mathematical induction

More details are in the handout.

Extension

12	0	12	24	36	48	60	72	84	96	108	120	132	144
11	0	11	22	33	44	55	66	77	88	99	110	121	132
10	0	10	20	30	40	50	60	70	80	90	100	110	120
9	0	9	18	27	36	45	54	63	72	81	90	99	108
8	0	8	16	24	32	40	48	56	64	72	80	88	96
7	0	7	14	21	28	35	42	49	56	63	70	77	84
6	0	6	12	18	24	30	36	42	48	54	60	66	72
5	0	5	10	15	20	25	30	35	40	45	50	55	60
4	0	4	8	12	16	20	24	28	32	36	40	44	48
3	0	3	6	9	12	15	18	21	24	27	30	33	36
2	0	2	4	6	8	10	12	14	16	18	20	22	24
1	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9	10	11	12

-6 -5 -4 -3 -2 -1

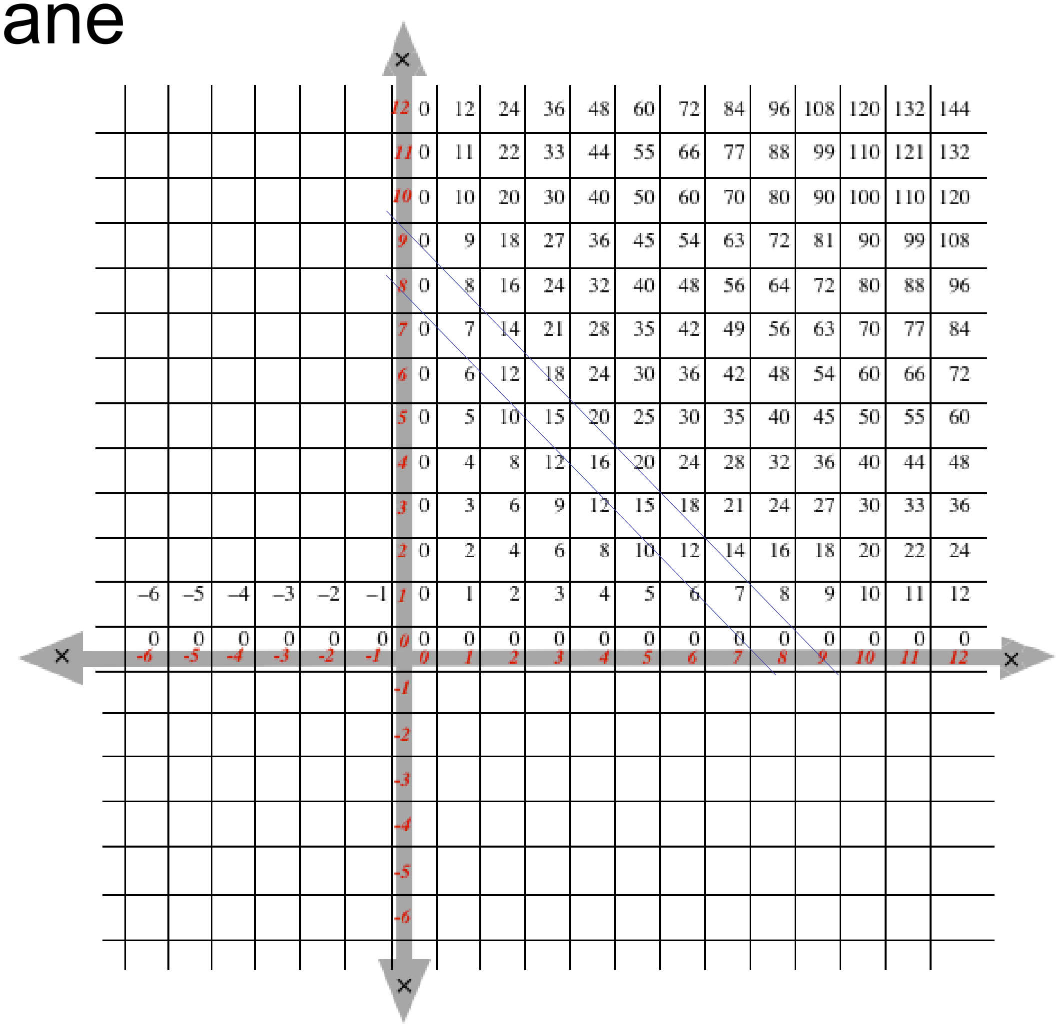
-1 -2 -3 -4 -5 -6

Extension: Tables of Arithmetic

- Oriented grid allows for connections with coordinate plane

All pairs of numbers whose sum is 8 lie along the graph of

$$x + y = 8$$



Extension: Tables of Arithmetic

- Oriented grid allows for natural extension to negatives

Students recognize and extend patterns that work in Quadrant I into the others

	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
12					0	12	24	36	48	60	72	84	96	108	120	132	144
11					0	11	22	33	44	55	66	77	88	99	110	121	132
10					0	10	20	30	40	50	60	70	80	90	100	110	120
9					0	9	18	27	36	45	54	63	72	81	90	99	108
8					0	8	16	24	32	40	48	56	64	72	80	88	96
7					0	7	14	21	28	35	42	49	56	63	70	77	84
6					0	6	12	18	24	30	36	42	48	54	60	66	72
5					0	5	10	15	20	25	30	35	40	45	50	55	60
4					0	4	8	12	16	20	24	28	32	36	40	44	48
3					0	3	6	9	12	15	18	21	24	27	30	33	36
2					0	2	4	6	8	10	12	14	16	18	20	22	24
1	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1		3			0	-1											
-2		6			0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20	-22	-24
-3		9			0	-3											
-4		12			0	-4											

Extension: Tables of Arithmetic

- Students discover properties of multiplication

  Experience with the number system comes before learning the basic moves of arithmetic (the any-order, any grouping properties).

- The extension isn't arbitrary!

  If the rules that work for positive numbers continue to work for negatives, then the product of two negative numbers must be positive.

Extension: Exponents

- Correct extension suggested by tables

  If the rule in the table continues to work, what is 2^0? What is 2^{-3}?

- Correct extension suggested by laws of exponents

  If $2^3 \cdot 2^0 = 2^3$, what should 2^0 be?

  What should $2^5 \cdot 2^{-3}$ equal?

- Students learn how to define rational exponents by relating them to geometric sequences

  $27^{2/3}$ is two-thirds of the way from 1 to 27…

x	$f(x) = 2^x$
3	8
2	4
1	2
0	
-1	

0, 9, 18, 27, 36, …

1, 3, 9, 27, 81, …

$9 = 27^{2/3}$

$81 = 27^{4/3}$

Extensions in *The CME Project*

- Complex numbers… If $a + b = 10$ and $ab = 29$, what is $a^2 + b^2$?
- Trigonometric functions… How should $\sin 150^\circ$ be defined?

Not every rule carries into an extension…

- Complex numbers don't follow $\sqrt{m} \cdot \sqrt{n} = \sqrt{mn}$
- Matrix multiplication isn't commutative
- If $x > y$, it may not be true that $\sin x > \sin y$

… but important rules often do

- Any nonzero complex number has a reciprocal
- Concepts of identity and inverse are important for matrices
- $\sin^2 x + \cos^2 x = 1$ even under the extended domain

Encapsulation: Guess-Check-Generalize

- Finding equations for geometric objects
- Building functions to model situations
- Refining calculations into algorithms

Guess-Check-Generalize

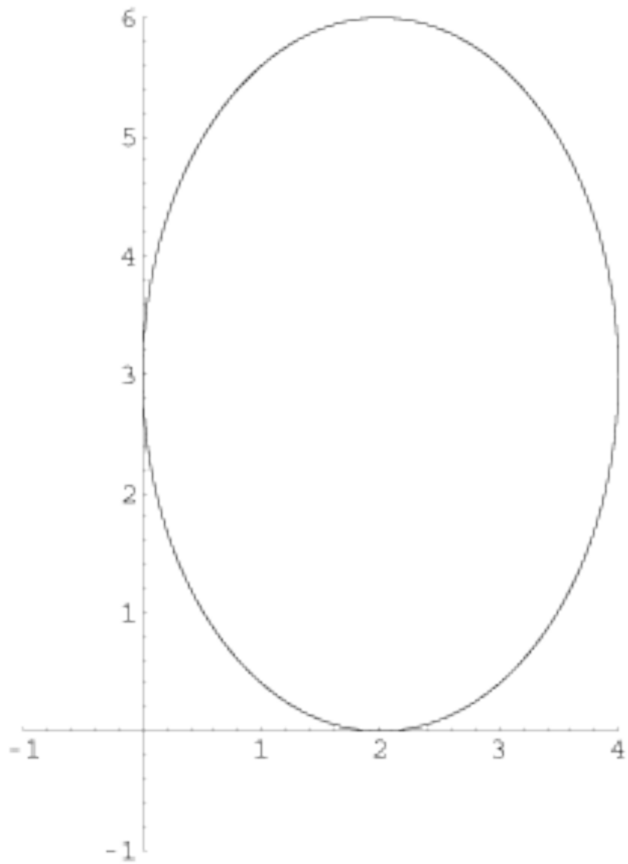
- Take a guess, any guess
- See if it works, keeping track of all steps
- Try another guess to get the pattern
- Finally, use a variable for the guess

Point-Tester

$$9x^2 - 36x + 4y^2 - 24y + 36 = 0$$

$$\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1$$

Point-Tester



$$\frac{(x-2)^2}{4}+\frac{(y-3)^2}{9}=1$$

Is the point (3, 5.6) on this ellipse?

Equations of Lines

- Slope-Intercept Form: $y = mx + b$
- Point-Slope Form: $(y - k) = m(x - h)$
- Standard Form: $Ax + By = C$

Point-Tester Approach

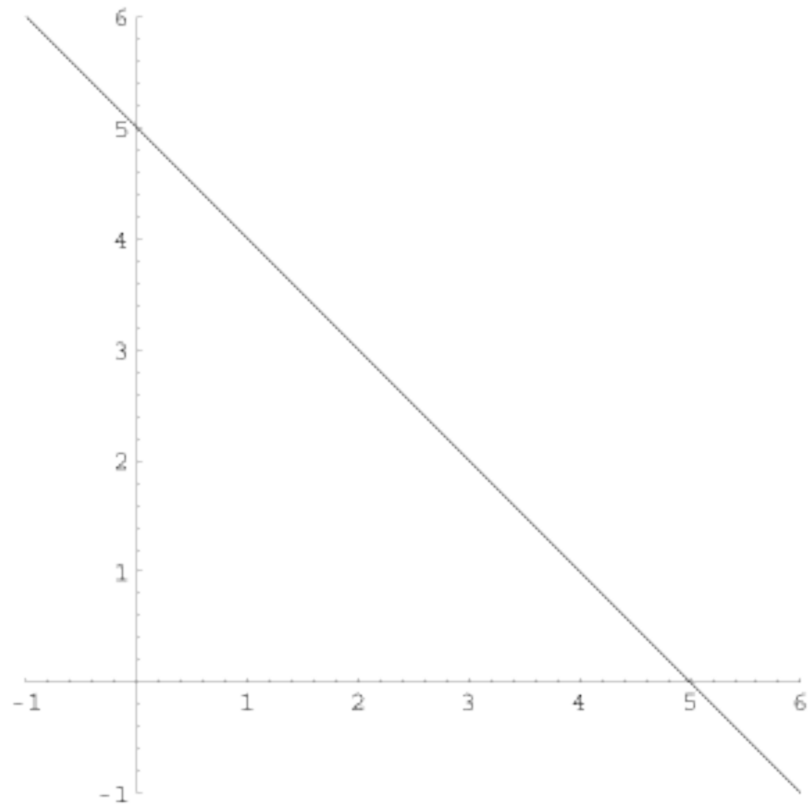
- Equations are *point-testers*
- Works for any two-variable equation!

Point-Tester Approach

Is (1,4) on the graph of $x + y = 5$?

$1 + 4 = 5$

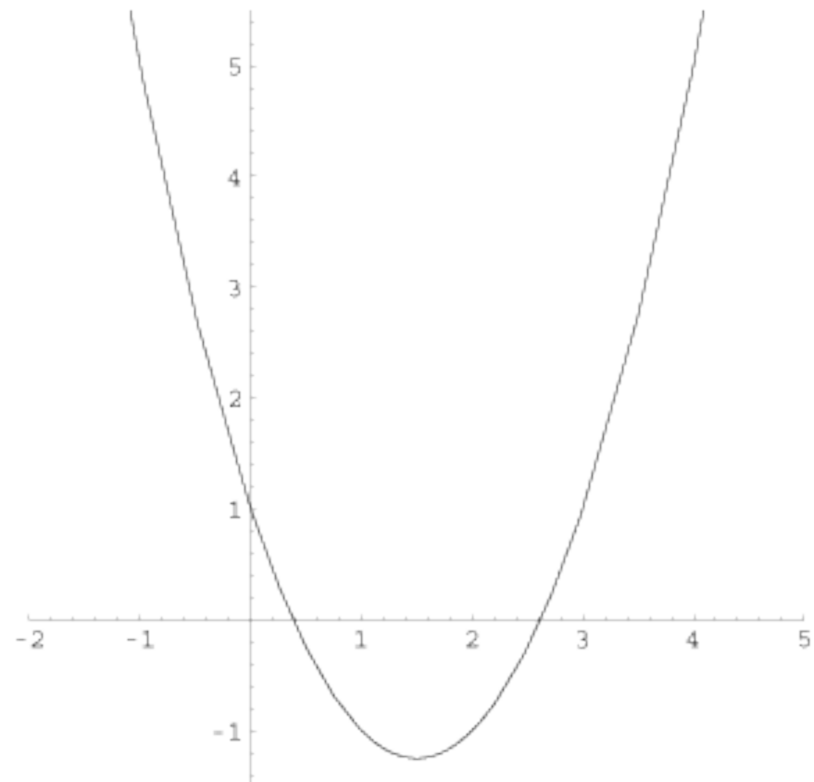
$5 = 5$ ✓



Point-Tester Approach

Is (-1,5) on the graph of
$y = x^2 - 3x + 1$?

$5 = (-1)^2 - 3(-1) + 1$
$5 = 1 + 4 + 1$
$5 = 6$ ✗



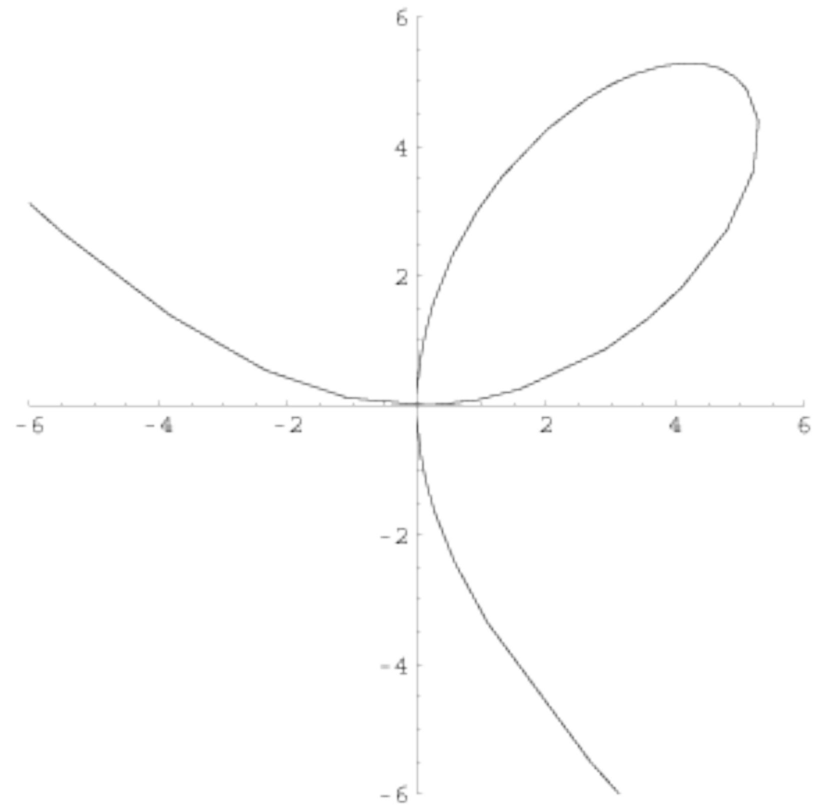
Point-Tester Approach

Is (5,5) on the graph of $x^3 + y^3 = 10xy$?

$5^3 + 5^3 = 10(5)(5)$

$125 + 125 = 250$

$250 = 250$ ✓



Finding an Equation of a Line

- Find the equation of a horizontal line through (5,1)
  - draw the line
  - locate a few points…what do they have in common?
  - is the y-coordinate 1?
  - is it true for (x,y) that $y = 1$?
- The equation of the line is $y = 1$.

Defining Slope

- Definition:
  the **slope** between two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is the rise over run:
  $$\frac{y_2 - y_1}{x_2 - x_1}$$
- Assumption: three points *A, B,* and *C* are **collinear** if and only if $m(A,B) = m(B,C)$
- Corollary: the slope between any two points on a line is **constant**.

Finding an Equation of a Line

Is the point (7,6)
on the line through
(4,1) and (5,3)?

Finding an Equation of a Line

Is the point (7,6) on the line through (4,1) and (5,3)?

$$\frac{3-1}{5-4} = \frac{2}{1} = 2 \qquad \frac{6-1}{7-4} = \frac{5}{3}$$

The slopes aren't the same, so (7,6) is not on the line.

Finding an Equation of a Line

Is the point (x,y)
on the line through
(4,1) and (5,3)?

Finding an Equation of a Line

Is the point (x,y) on the line through (4,1) and (5,3)?

$$\frac{3-1}{5-4}=\frac{2}{1}=2 \qquad \frac{y-1}{x-4}=2$$

so (x,y) is on the line if

$y - 1 = 2(x - 4)$

Working Toward General Forms

“One student suggested that once you find the slope (say, $^1/_2$), you could write $y = {}^1/_2x$, [but] she didn't know what to do with that…. I [reminded] the students [about] equations as point-testers and asked her what we might do from here.

Working Toward General Forms

She suggested plugging the point in for x and y. WOW!

I said, OK, but it doesn't satisfy the equation, and it has to, so what might we do? She suggested finding an adjustment amount to make it work. BINGO!

Working Toward General Forms

I got so excited, the students were very concerned! I've never had a *student* come up with how to use slope-intercept form to find the equation of a line before—all I've gotten [were] blank stares!"

- *Annette Roskam*
Rice Lake High School
Rice Lake, Wisconsin

Other Examples of Encapsulation

- Solving word problems
- Building functions to model situations
- Monthly payments on a loan

Three Themes

PROOF

- Proof as generalized instance
- Proof in geometry
- Proof by mathematical induction

Proof as generalized instance

- Find a polynomial that agrees with the table at right:

x	$s(x)$
1	0
3	0

Proof as generalized instance

$s(x) = (x-1)(x-3)$

x	$s(x)$
1	0
3	0

Proof as generalized instance

Find a polynomial that agrees with the table at right:

x	$j(x)$
1	0
3	0
6	60

Proof as generalized instance

$j(x)=M(x-1)(x-3)$

for some constant M.

x	$j(x)$
1	0
3	0
6	60

Proof as generalized instance

60 = M(6-1)(6-3)

so

4 = M.

Thus

$$j(x) = 4(x-1)(x-3)$$
$$= 4x^2 - 16x + 12$$

agrees with the table.

x	$j(x)$
1	0
3	0
6	60

A (slightly) more complicated example

How might you find a polynomial that agrees with this table?

x	$f(x)$
1	20
3	12
6	60

A (slightly) more complicated example

x	$f(x)$
1	20
3	12
6	60

=

x	$g(x)$
1	20
3	0
6	0

+

x	$h(x)$
1	0
3	12
6	0

+

x	$j(x)$
1	0
3	0
6	60

Let's look at $g(x)$

x	$g(x)$
1	20
3	0
6	0

$$g(x) = A(x-3)(x-6)$$

$$g(1) = A(1-3)(1-6)$$

$$20 = A(-2)(-5)$$

$$A = 2$$

$$g(x) = 2x^2 - 18x + 36$$

Similarly…

x	$h(x)$
1	0
3	12
6	0

$$h(x) = B(x-1)(x-6)$$

$$h(3) = B(3-1)(3-6)$$

$$12 = B(2)(-3)$$

$$B = -2$$

$$h(x) = -2x^2 + 14x - 12$$

Adding this up…

x	$f(x)$
1	20
3	12
6	60

=

x	$g(x)$
1	20
3	0
6	0

$g(x) = 2x^2 - 18x + 36$

+

x	$h(x)$
1	0
3	12
6	0

$h(x) = -2x^2 + 14x - 12$

+

x	$j(x)$
1	0
3	0
6	60

$j(x) = 4x^2 - 16x + 12$

$$f(x) = 4x^2 - 20x + 36$$

And finding *f*(*x*)…

x	$f(x)$
1	20
3	12
6	60

=

x	$g(x)$
1	20
3	0
6	0

+

x	$h(x)$
1	0
3	12
6	0

+

x	$j(x)$
1	0
3	0
6	60

$$f(x) = 4x^2 - 20x + 36$$

A teacher's view

"I didn't even really remember what Lagrange Interpolation was, so I didn't really want to bother…But I couldn't believe the connections my students made when they started working on it. I was floored – they made connections that they had never made before. Light bulbs were going off everywhere….They understood how to add functions, and why you might want to. They understood that functions are things you *can* add. They saw some value to factoring, because they really understood the relationship between factors and roots. And what surprised me most of all was how much they loved solving the problems – because they were good at it."

Chris Martino, Arlington High School

Why Bother with Proof?

Because of tradition, culture and necessity:

mathematicians are great experimenters.

They

- perform thought experiments,
- build concrete and mental models,
- gather and analyze data,
- make conjectures.

But to be able to draw conclusions they must rely on ***deduction*** and ***proof***.

Intuition to Formalization

How do you come up with the *idea* for a proof?

In ***The CME Project*** we recognize that finding the sequence of ideas that constitute a proof is the difficult task. The written proof is just a record. Students learn three techniques that help them develop the ideas:

- the visual scan;
- the flow chart;
- the reverse list.

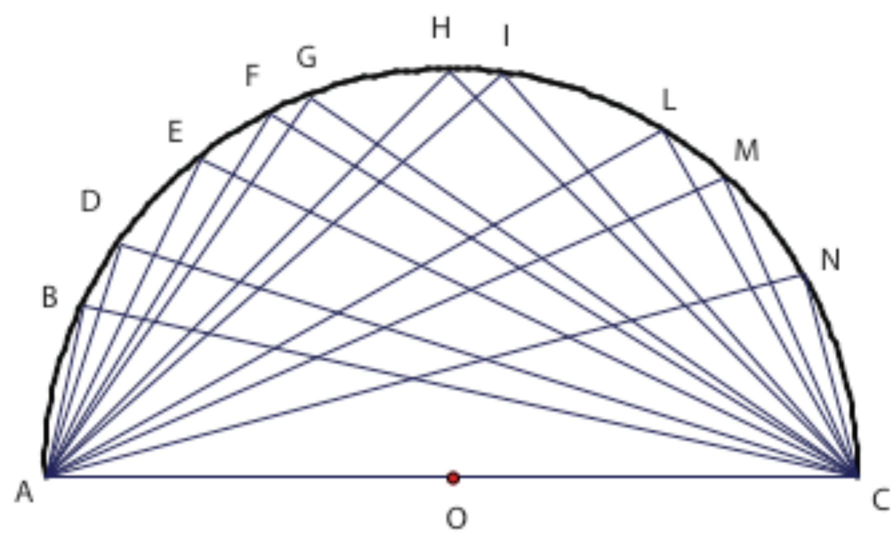
Coming Up with a Proof:
Angles Inscribed in a Semicircle

Draw a semicircle and inscribe 10 angles in it.

a) Measure the angles in degrees.

b) Can you find invariants? State your conjecture.

c) Prove your conjecture.



Coming Up with a Proof

Solution:

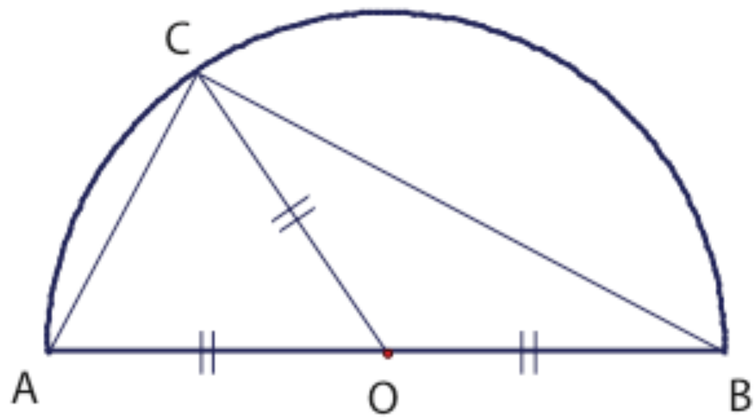
a) Your measures will be all close to 90˚.

b) Some possible conjectures are:

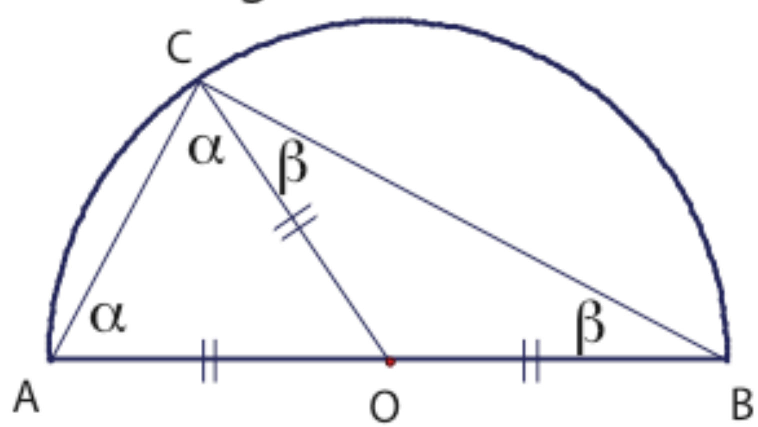
- All the angles inscribed in my semicircle are congruent.
- All the angles inscribed in my semicircle are right angles.
- All the angles inscribed in any semicircle are right angles.

Coming Up with a Proof:
Angles Inscribed in a Semicircle

c) Visual scan: Add a radius and mark radii.



Flow chart: Notice congruent angles in isosceles triangles



Reverse list: Need to prove $\alpha + \beta = 90^\circ$

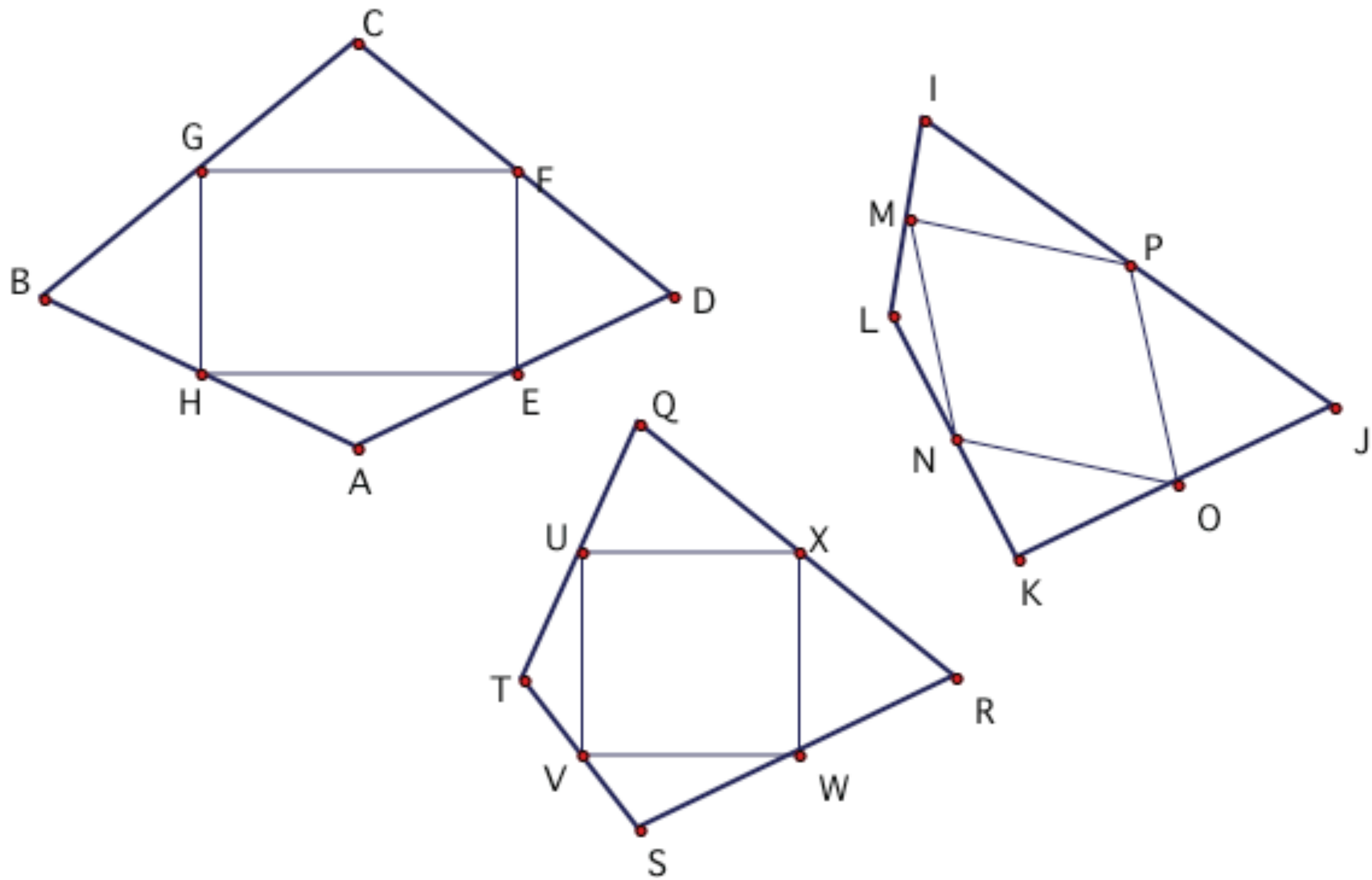
Can use: Sum of angles of a triangle is 180°

So Why Use Proof?

- To be able to ***convince*** yourself and others.
- To understand ***why***.
- To be able to ***explain*** a new observation.
- For ***generality***.
- To ***check*** your measurements.
- Because it is a distinguishing feature of ***mathematical research***.

Proof as a Research Technique: Midpoints of a Quadrilateral

Students are motivated to continue experimenting and coming up with conjectures about when the parallelogram is a rhombus, a rectangle, a square…



More Details…

- Look for us at NCTM Regional in Chicago (Computer Algebra Systems)
- Handouts
- Web: http:/www.edc.org/cmeproject

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