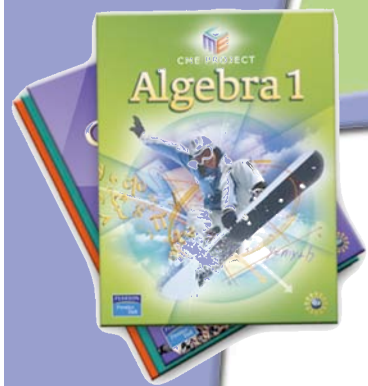
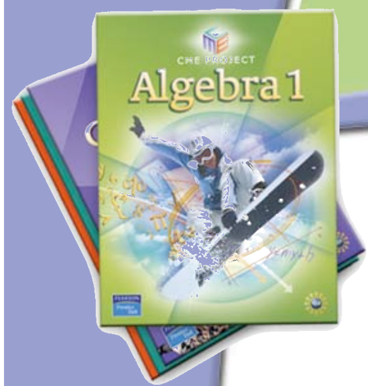


Developing Proof Throughout High School Mathematics

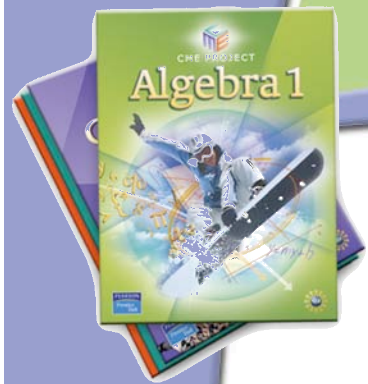
Kevin Waterman
Anna Baccaglini-Frank
Doreen Kilday
Education Development Center

EDC



Experimenting

Recasting Basic Arithmetic Tables



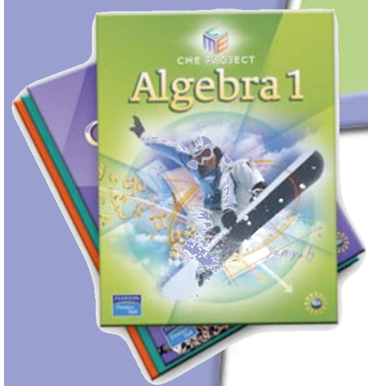
Experimenting

Addition Table

	0	1	2	3	4	5	6	7	8	9	10	11	12
12	12	13	14		16	17	18	19	20	21	22	23	24
11	11	12	13			16	17	18	19	20	21	22	23
10	10	11	12		14		16	17	18	19	20	21	22
9	9	10	11		13	14		16	17	18	19	20	21
8	8	9	10		12	13	14		16	17	18	19	20
7	7	8	9		11	12	13	14		16	17	18	19
6	6	7	8		10	11	12	13	14		16	17	18
5	5	6	7		9	10	11	12	13	14		16	17
4	4	5	6		8	9	10	11	12	13	14		16
3	3	4	5		7	8	9	10	11	12	13	14	
2	2	3	4		6	7	8	9	10	11	12		14
1	1	2	3		5	6	7	8	9	10		12	13
0	0	1	2		4	5	6	7	8		10	11	12

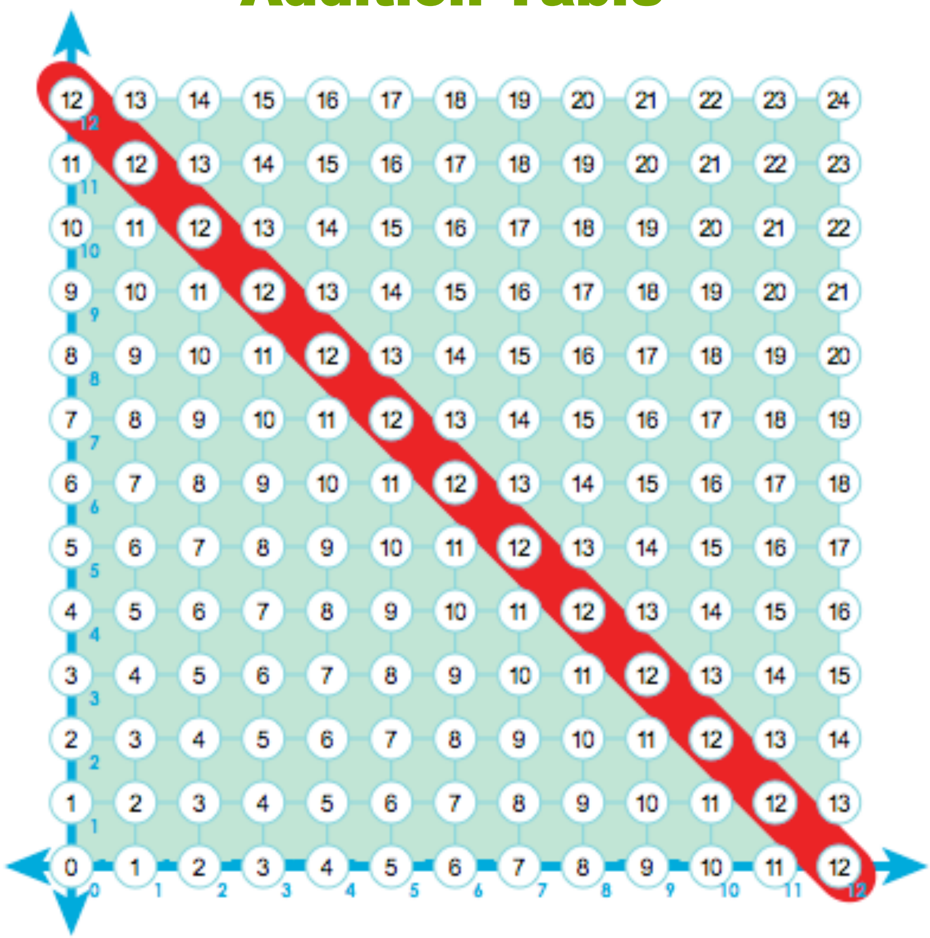
Multiplication Table

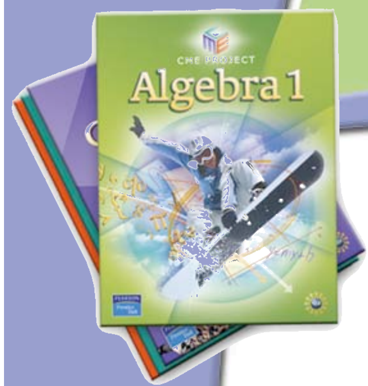
	0	1	2	3	4	5	6	7	8	9	10	11	12
12		12	24	36	48	60	72		96	108	120	132	144
11	0		22	33	44	55		77	88	99	110	121	132
10	0	10		30			60	70	80	90	100	110	120
9	0	9	18			45	54	63	72	81	90	99	108
8	0	8	16			40	48	56	64	72	80	88	96
7	0	7		21	28		42	49	56	63	70	77	84
6	0		12	18	24	30		42	48	54	60	66	72
5		5	10	15	20	25	30		40	45	50	55	60
4	0	4	8	12	16	20	24	28		36	40	44	48
3	0	3	6	9	12	15	18	21	24		30	33	36
2	0	2	4	6	8	10	12	14	16	18		22	24
1	0	1	2	3	4	5	6	7	8	9	10		12
0	0	0	0	0	0	0	0	0	0	0	0	0	



Experimenting

Addition Table





Experimenting

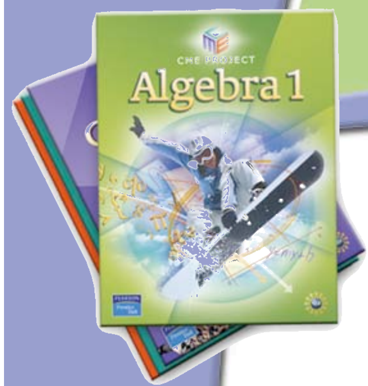
Multiplication Table

0	12	24	36	48	60	72	84	96	108	120	132	144
0	11	22	33	44	55	66	77	88	99	110	121	132
0	10	20	30	40	50	60	70	80	90	100	110	120
0	9	18	27	36	45	54	63	72	81	90	99	108
0	8	16	24	32	40	48	56	64	72	80	88	96
0	7	14	21	28	35	42	49	56	63	70	77	84
0	6	12	18	24	30	36	42	48	54	60	66	72
0	5	10	15	20	25	30	35	40	45	50	55	60
0	4	8	12	16	20	24	28	32	36	40	44	48
0	3	6	9	12	15	18	21	24	27	30	33	36
0	2	4	6	8	10	12	14	16	18	20	22	24
0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0

Experimenting

Conjecture:

The maximum product of two numbers whose sum is fixed occurs when the two numbers are equal.



Experimenting

Prove This Identity:

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$$

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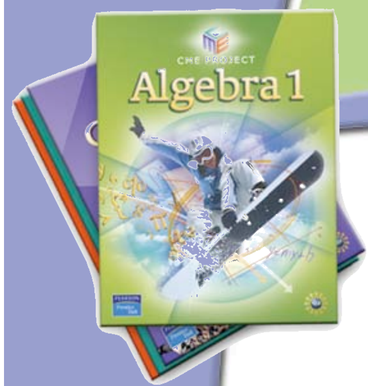


Experimenting

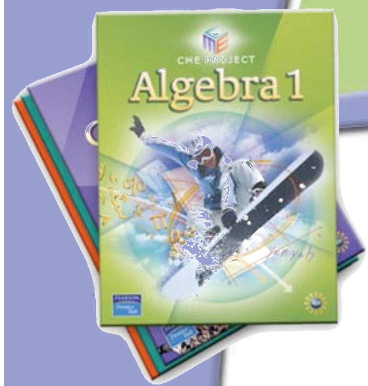
Prove The Identity Algebraically

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$$

$$\frac{a^2+2ab+b^2}{4} - \frac{a^2-2ab+b^2}{4} = ab$$

$$\frac{a^2+2ab+b^2-a^2+2ab-b^2}{4} = ab$$

$$\frac{4ab}{4} = ab$$



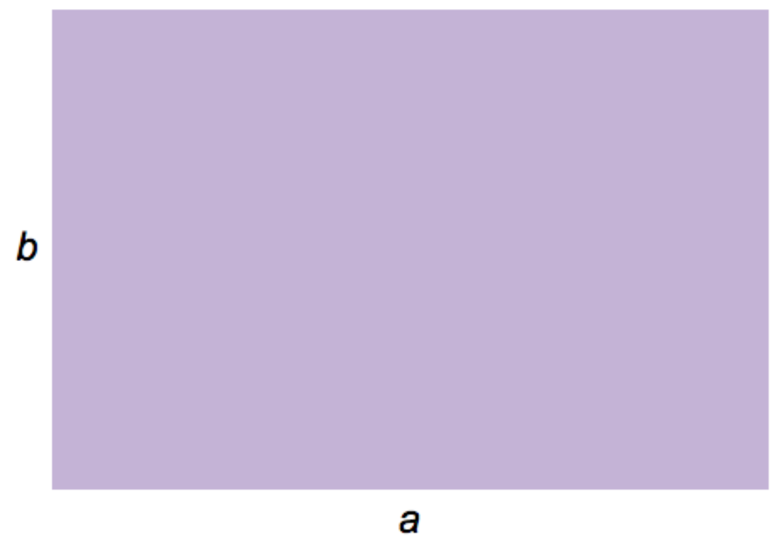
Experimenting

Prove This Identity:

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$$

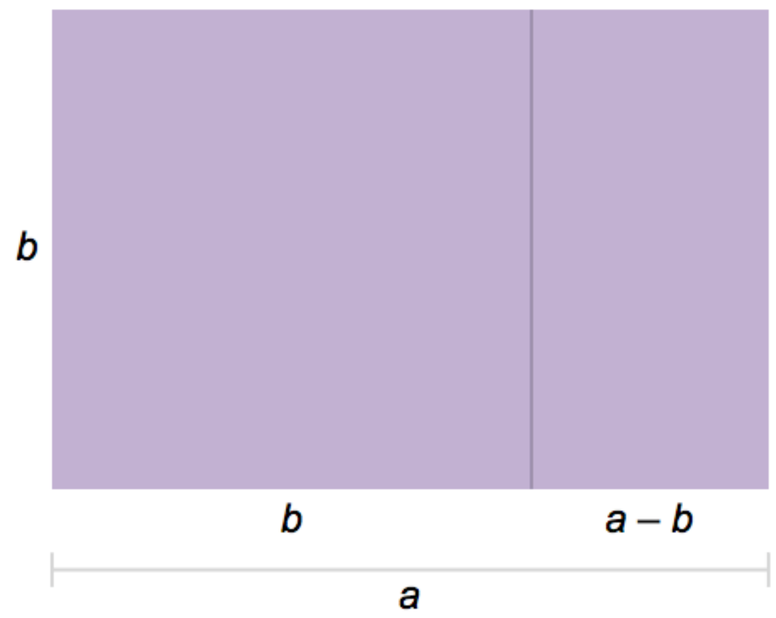
Experimenting

Prove The Identity Geometrically



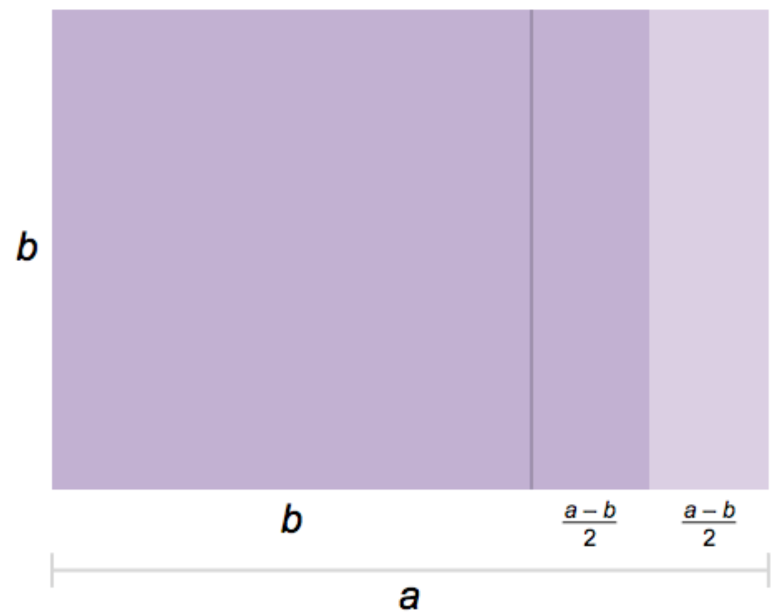
Experimenting

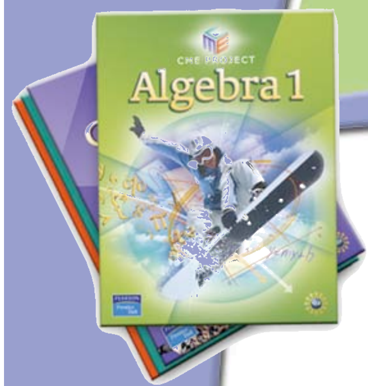
Prove The Identity Geometrically



Experimenting

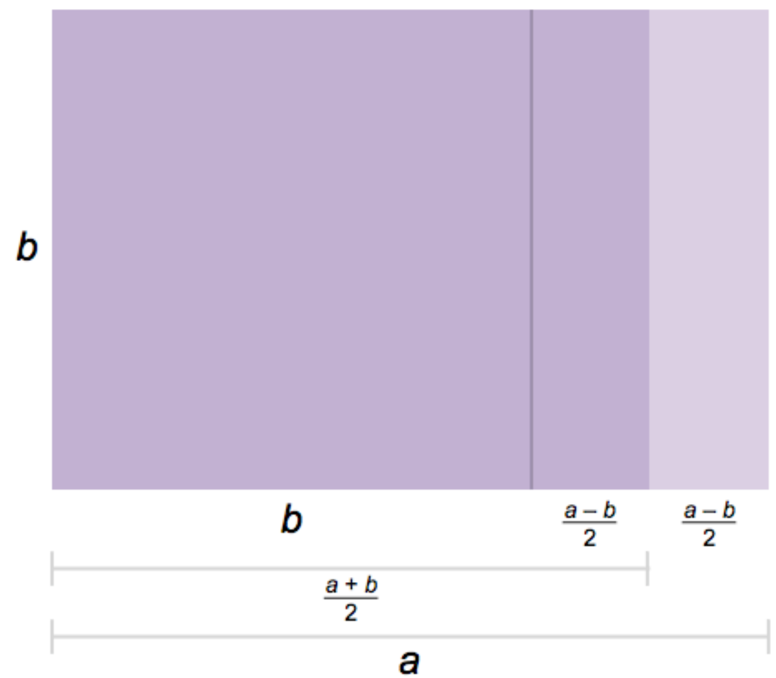
Prove The Identity Geometrically





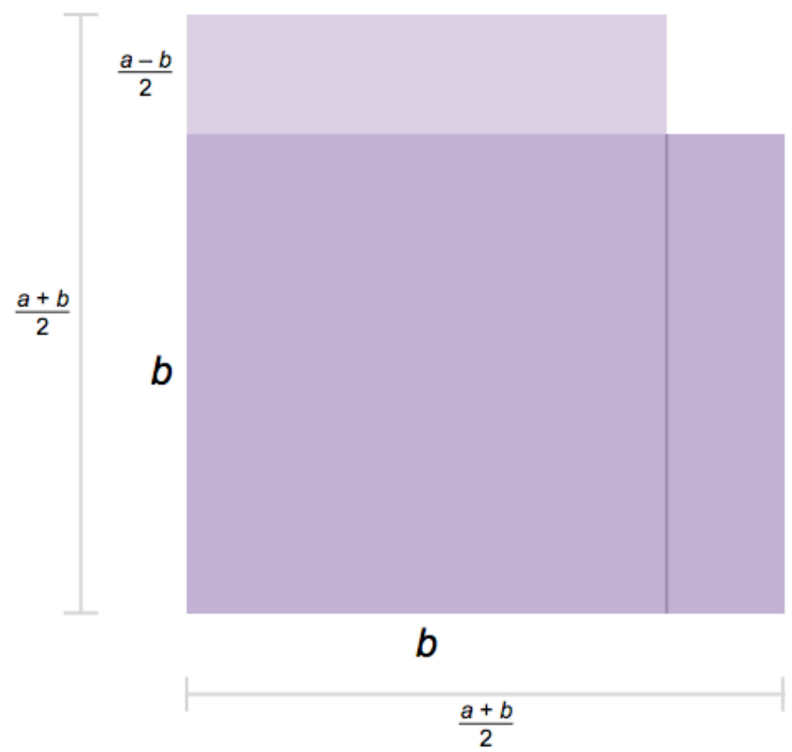
Experimenting

Prove The Identity Geometrically



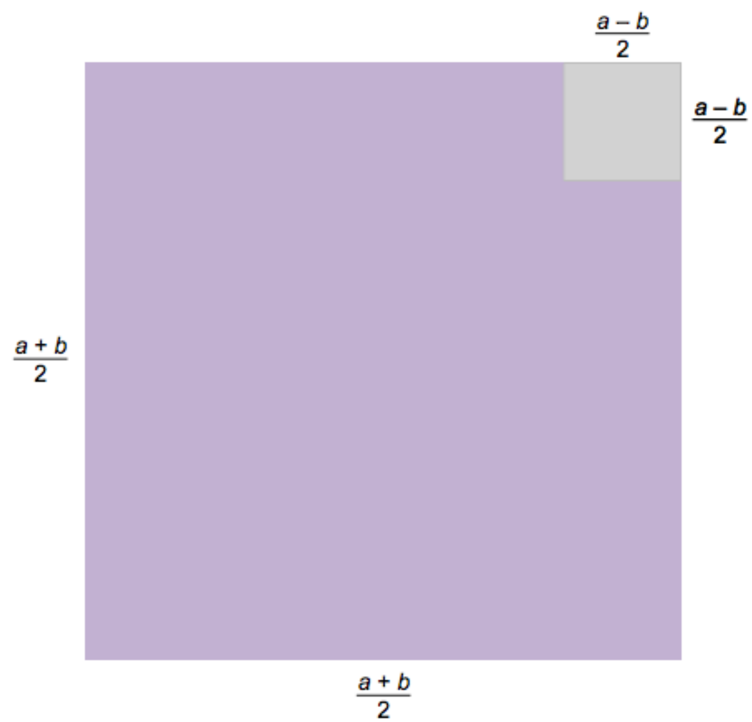
Experimenting

Prove The Identity Geometrically



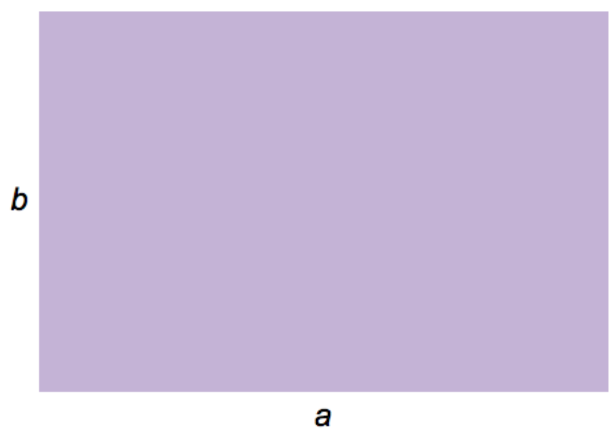
Experimenting

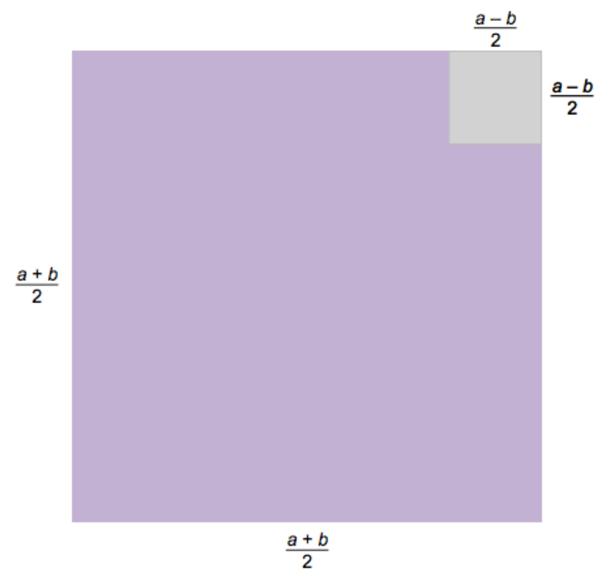
Prove The Identity Geometrically



Experimenting

Prove The Identity Geometrically





Area = ab

$$\text{Area} = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

Experimenting

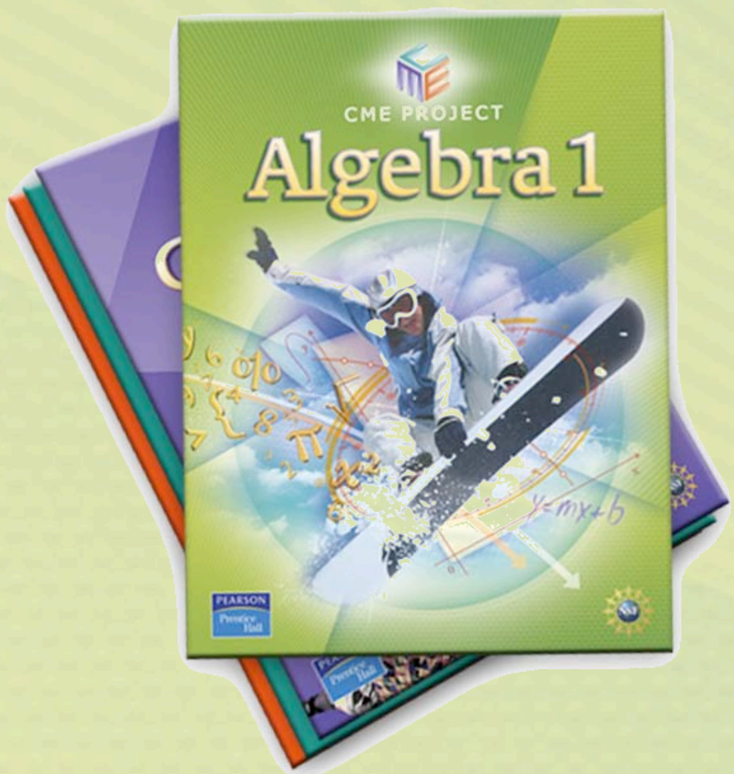
Theorem: The rectangle of perimeter P with maximum area is the square with side length $\frac{P}{4}$.

Theorem: The arithmetic mean of two positive real numbers is less than or equal to their geometric mean:

$$\frac{a+b}{2} \leq \sqrt{ab}$$

What is the CME Project?

- A Brand New, Comprehensive, 4-year Curriculum
- NSF-funded
- Problem-Based, Student-Centered Approach
- "Traditional" Course Structure



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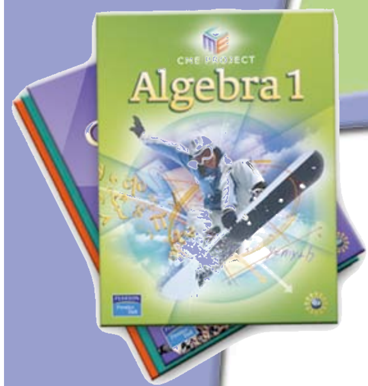

CME Project Overview

Contributors

- EDC's Center for Mathematics Education
- National Advisory Board
- Core Mathematical Consultants
- Teacher Advisory Board
- Field-Test Teachers

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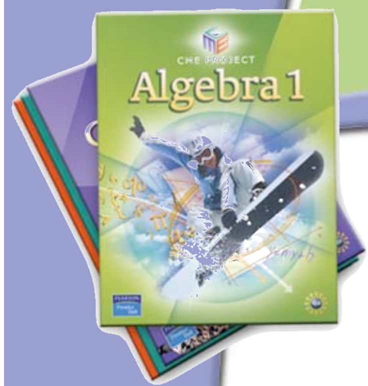
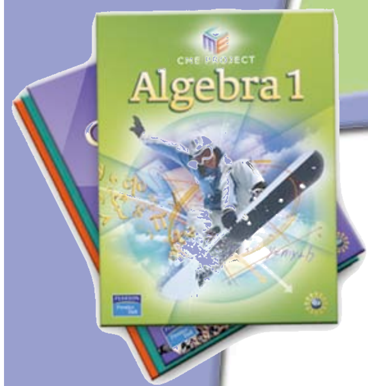


CME Project Overview

“Traditional” course structure: it’s familiar but different

- Structured around the sequence of Algebra 1, Geometry, Algebra 2, Precalculus
- Uses a variety of instructional approaches
- Focuses on particular mathematical habits
- Uses examples and contexts from many fields
- Organized around mathematical themes



CME Project Overview

CME Project audience:
the (large number of) teachers who…

- Want the familiar course structure
- Want a problem- and exploration-based program
- Want to bring activities to "closure"
- Want rigor and accessibility for all

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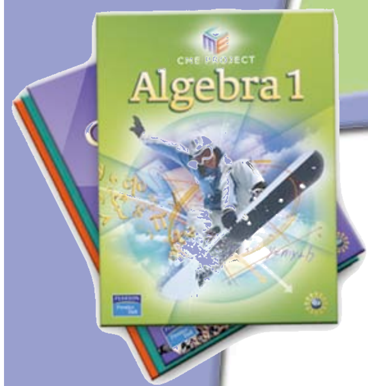
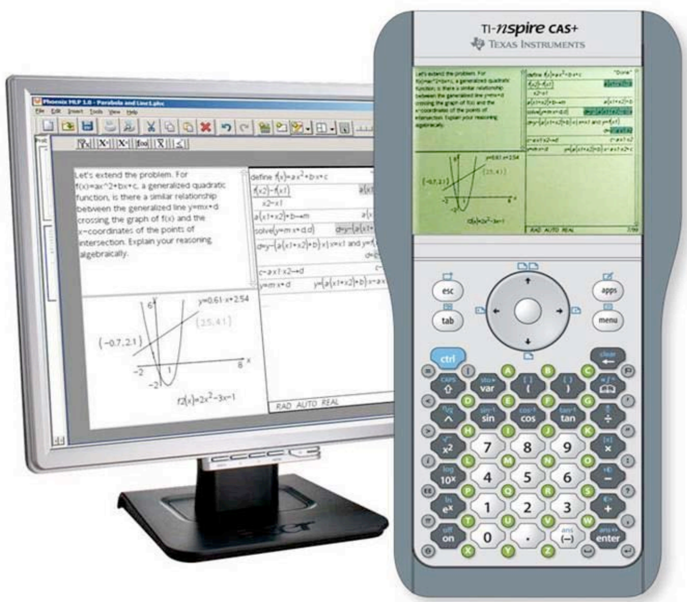


CME Project Overview

Relationship with Texas Instruments



CME Project makes essential use of technology:

- A “function-modeling” language (FML)
- A computer algebra system (CAS)
- An interactive geometry environment

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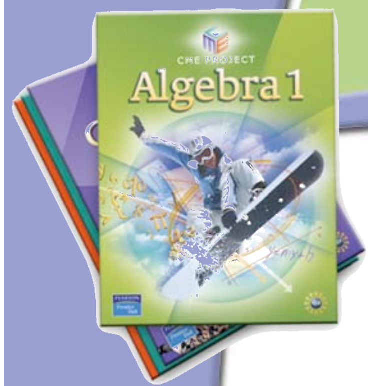

CME Project Overview

Fundamental Organizing Principle

The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more importantly, from developing the ways of thinking—the *habits of mind*—used to create the results.

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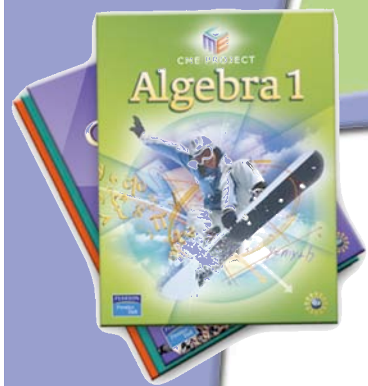
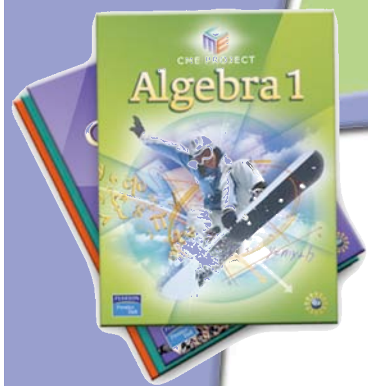


The Role of Proof

- As a method for establishing logical connections (and hence certainty)
- As a means for obtaining “hidden” insights
- As a research technique



Our Approach to Proof

- Distinguish between conception and presentation
- Provide experience with specific examples prior to abstraction

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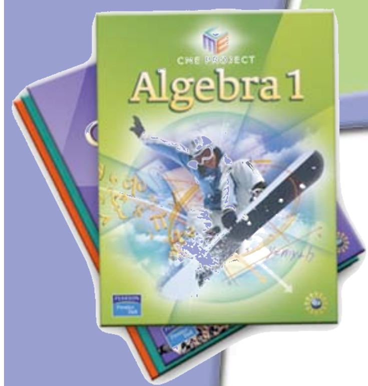
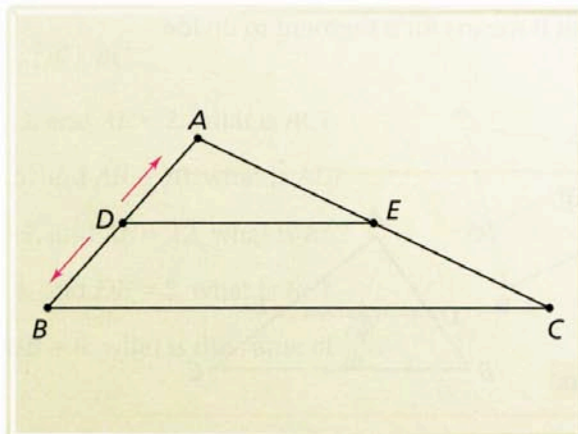


Experience Before Formality

Part 2 Splitting Two Sides of a Triangle

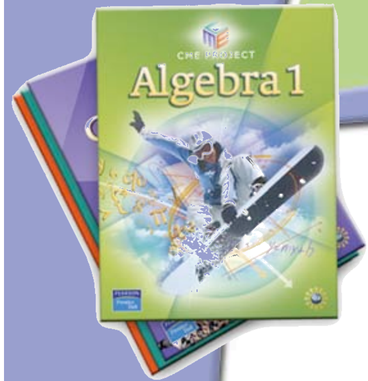
Use geometry software. Draw $\triangle ABC$. Place a point D anywhere on side $\overline{AB}$. Then construct a segment $\overline{DE}$ that is parallel to $\overline{BC}$.



Drag point D along $\overline{AB}$.

$\triangle ADE$ and $\triangle ABC$ are a pair of nested triangles.

6. Use the software to find the ratio $\frac{AD}{AB}$.
7. Find two other length ratios with the same value. Do all three ratios remain equal to each other when you drag point D along $\overline{AB}$?
8. As you drag D along $\overline{AB}$, describe what happens to the figure. Make a conjecture about the effect of $\overline{DE}$ being parallel to $\overline{BC}$.

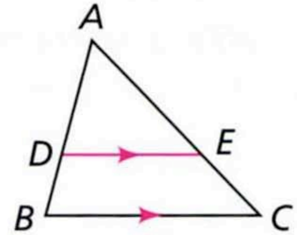


Side-Splitter Theorems

Theorem 4.1 *The Parallel Side-Splitter Theorem*

If a segment with endpoints on two sides of a triangle is parallel to the third side of the triangle, then

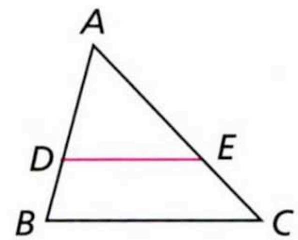
- the segment splits the sides it intersects proportionally
- the ratio of the length of the parallel side to the length of this segment is the common ratio



$$\frac{AB}{AD} = \frac{AC}{AE} = \frac{BC}{DE}$$

Theorem 4.2 *The Proportional Side-Splitter Theorem*

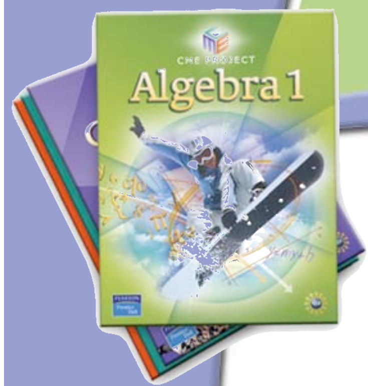
If a segment with endpoints on two sides of a triangle splits those sides proportionally, then the segment is parallel to the third side.



$$\frac{AB}{AD} = \frac{AC}{AE} \Rightarrow \overline{DE} \parallel \overline{BC}$$

Some Methods for Conceiving a Proof

- Visual Scan
- Flowchart
- Reverse List



The Reverse List Method

- Start with what you want to prove and move backward.
- Repeatedly ask questions:
  - What information do I need?
  - What strategy can I use to prove that?

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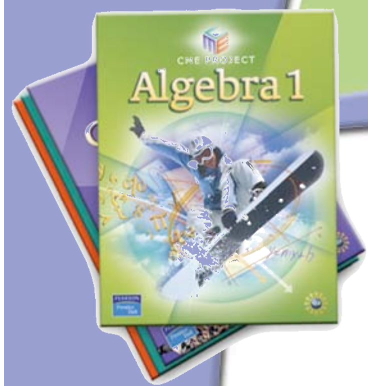
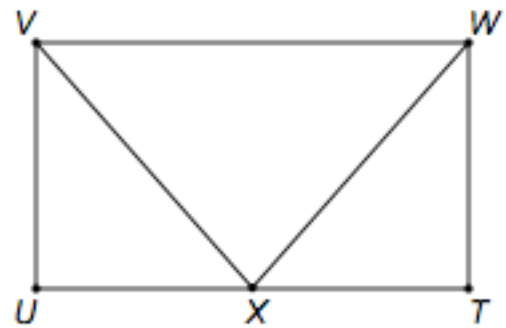


An Example of the Reverse List Method



Given: ***TUVW* is a rectangle**
X* is the midpoint of *TU

Prove: **Triangle *XWV* is isosceles**

Need: ΔXWV is isosceles

Use: a triangle is isosceles if two sides are congruent.

Need: $\overline{XW} \cong \overline{XV}$

Use: CPCTC

Need: Congruent Triangles

Use: SAS with ΔWXT and ΔVXU

Need: first side $\overline{TW} \cong \overline{UV}$

Use: opposite sides of a rectangle are congruent

Need: *TUVW* is a rectangle

Use: Given

Need: $\angle T \cong \angle U$

Use: all angles of a rectangle are congruent

Need: second side $\overline{TX} \cong \overline{UX}$

Use: The midpoint of a segment divides it into two congruent segments

Need: *X* is the midpoint of $\overline{TU}$

Use: Given

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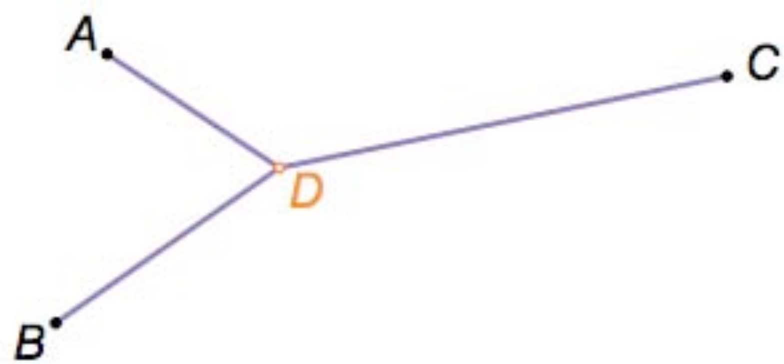
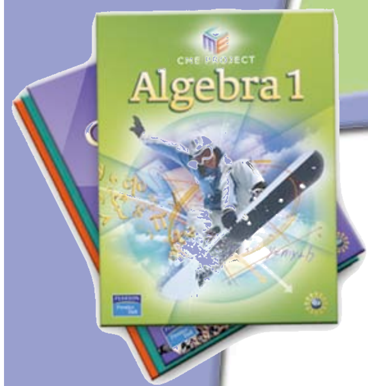
The Airport Problem

Let A, B, and C be the locations of three cities and let D be the location of a new airport serving them. Where should an airport be built that minimizes the sum of its distances to the three cities?



Investigate Using a Dynagraph

- Use the dynamic geometry software to mark the lengths of each segment and calculate s, the sum of these lengths.
- Then construct a segment $\overline{QP}$ of length s.
- Drag D to make a conjecture about choices of D that minimize s.
- Try your conjecture again after moving A, B, and/or C.



Airport Problem

Conjecture:

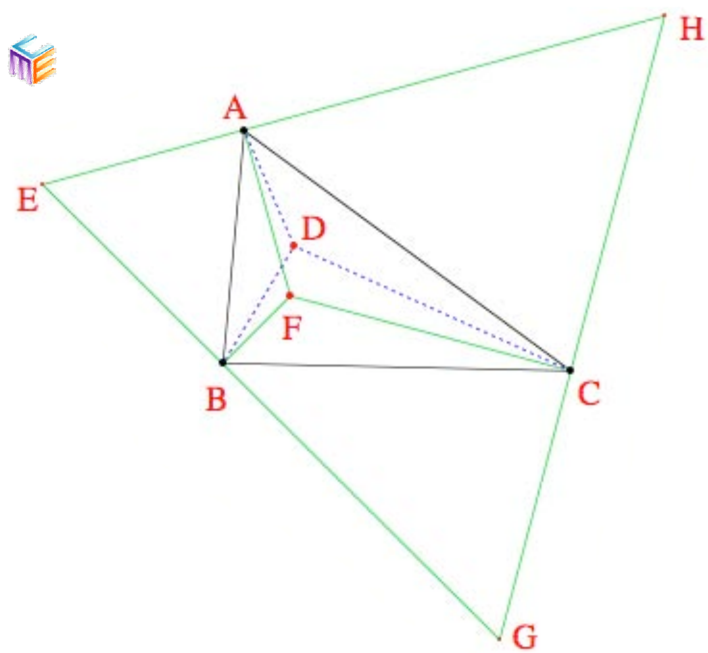
If the three cities are the vertices of an acute triangle, then the best place for the airport is where the roads form 120º angles with each other.

Proving the Airport Problem

- Let F be a point such that $m\angle AFB = m\angle BFC = m\angle CFA = 120°$.
- Construct
  EH ⊥ AF through A,
  EG ⊥ BF through B, and
  GH ⊥ CF through C.

Need to prove:

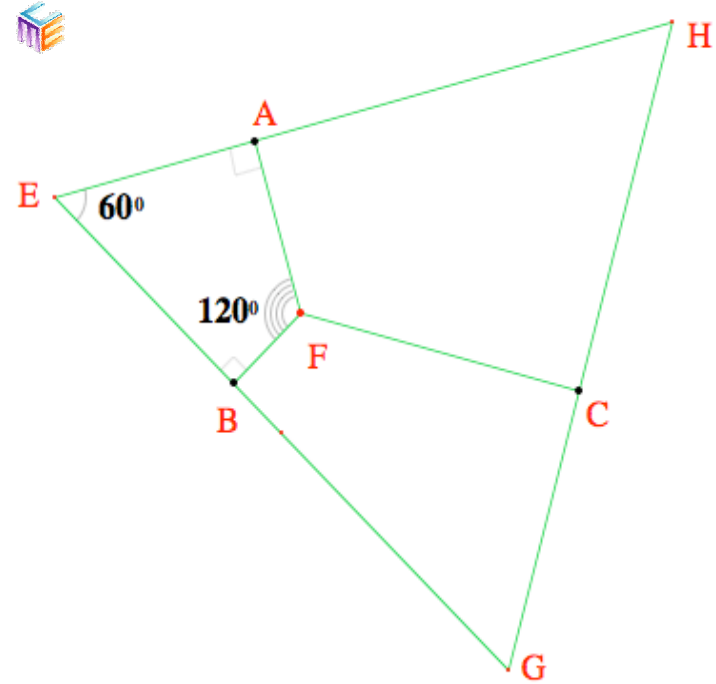
- ΔEHG is equilateral.
- The sum of the distances to A, B, C of any point D different from F is greater.

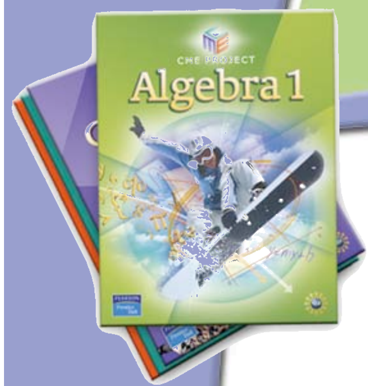


Proving the Airport Problem

Prove that $\triangle EHG$ is equilateral

- AEBF is a quadrilateral.
- $\angle EAF$ and $\angle EBF$ are right angles.
- $m\angle AFB = 120^\circ$.
- So $m\angle AEB = 540^\circ - 180^\circ - 180^\circ - 120^\circ = 60^\circ$.
- Similarly $m\angle EHG = m\angle HGE = 60^\circ$.





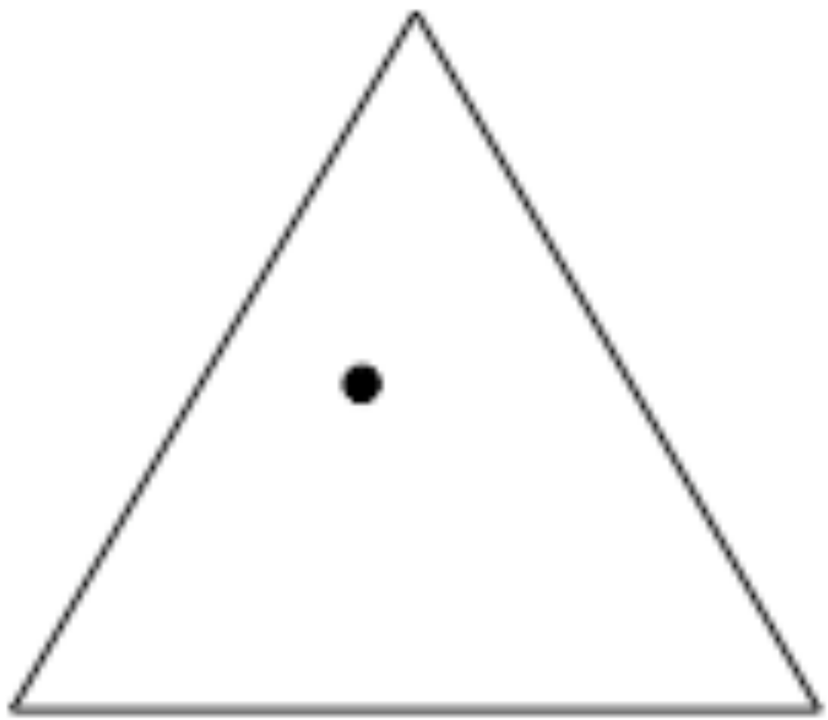
Proving the Airport Problem

Prove that the sum of the distances to *A*, *B*, *C* of any point *D* different from *F* is greater:

Theorem (Rich's Theorem):
In an equilateral triangle the sum of the distances of a point inside the triangle to the sides is constant.

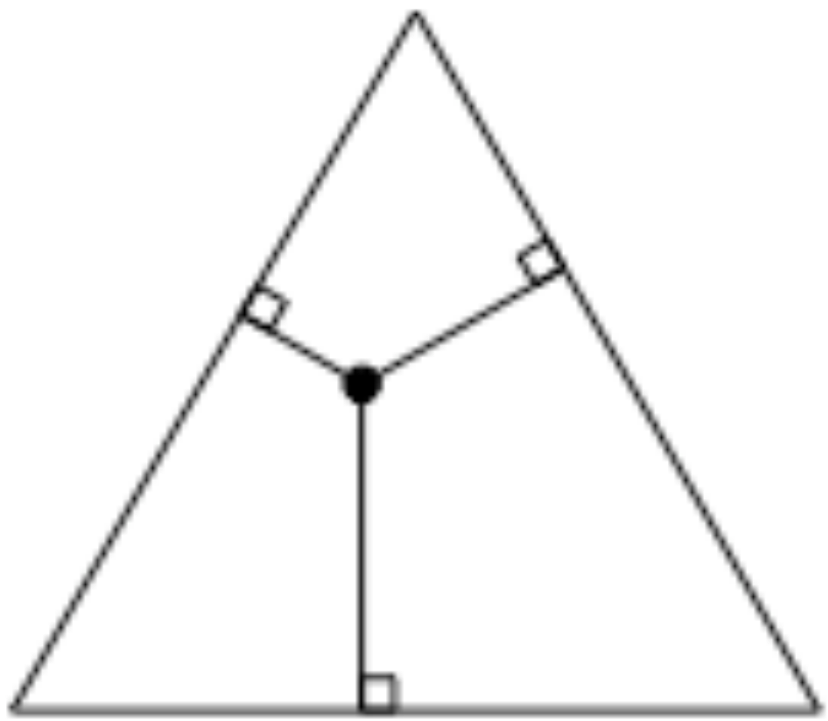
Proving the Airport Problem

Prove Rich’s Theorem



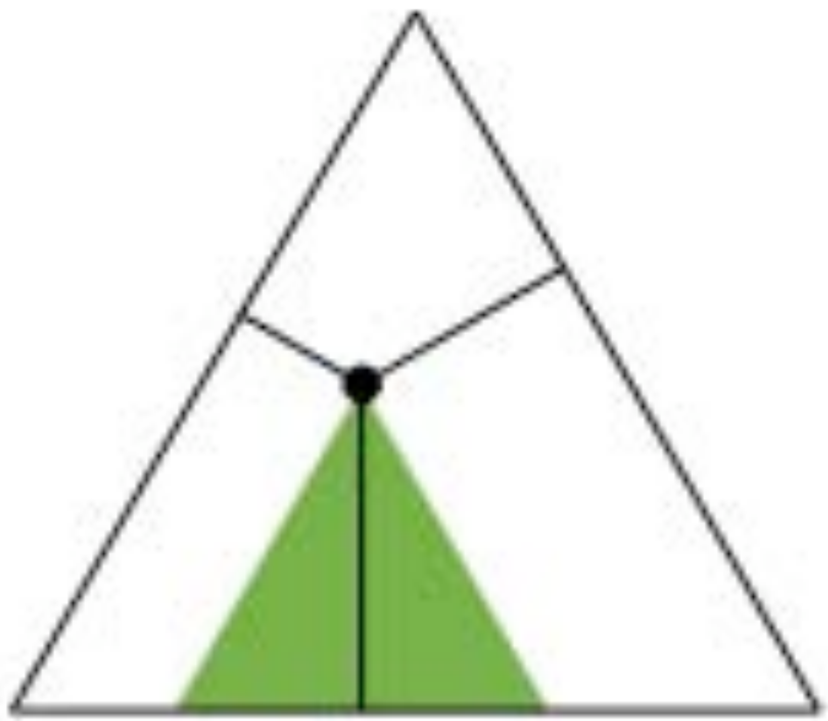
Proving the Airport Problem

Prove Rich's Theorem



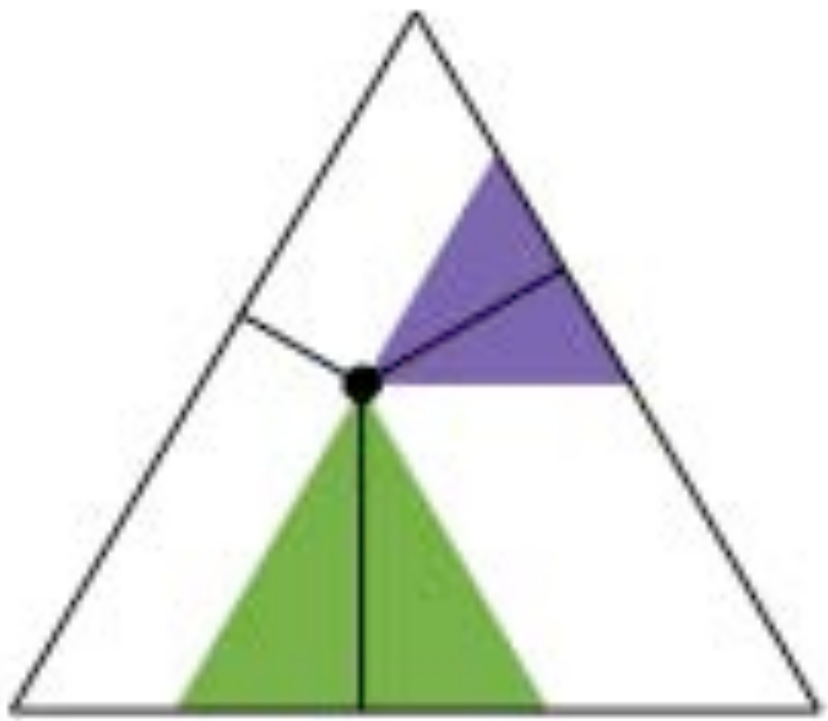
Proving the Airport Problem

Prove Rich's Theorem



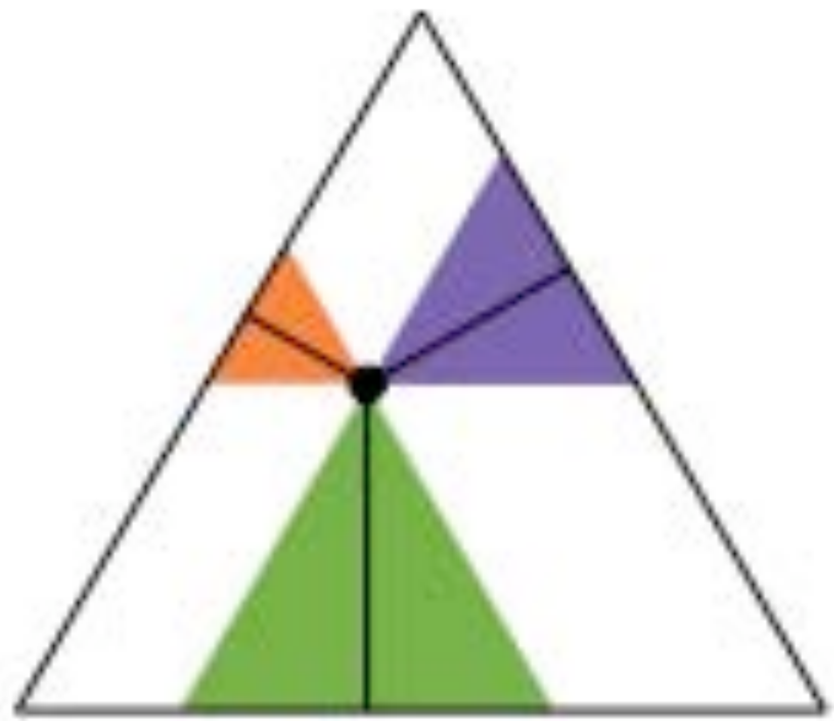
Proving the Airport Problem

Prove Rich's Theorem



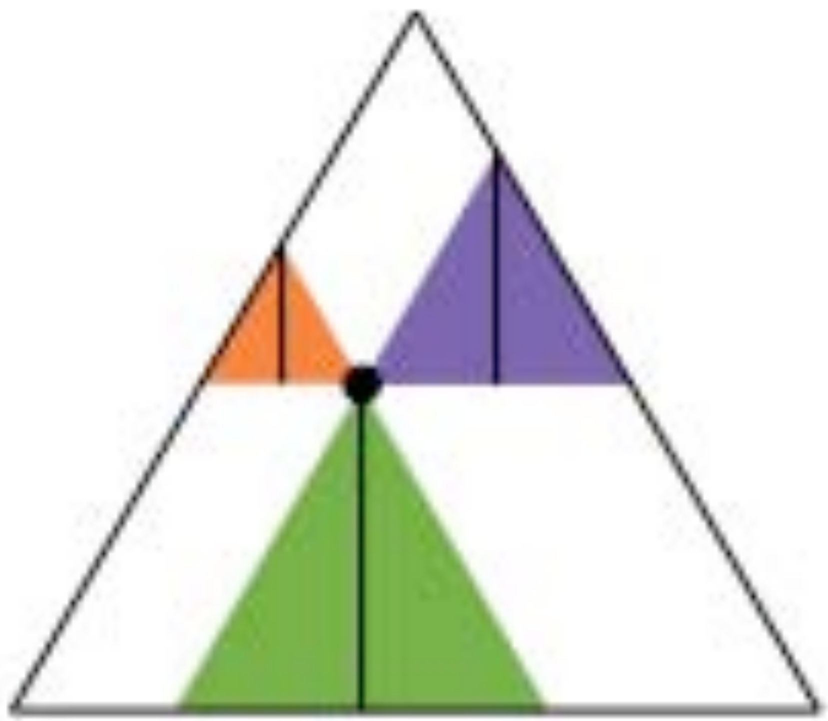
Proving the Airport Problem

Prove Rich's Theorem



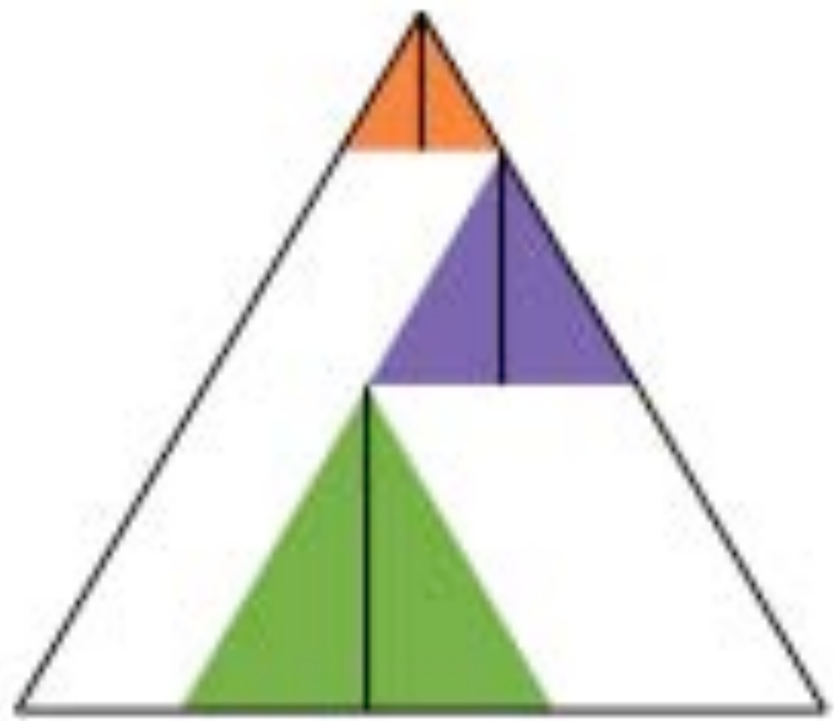
Proving the Airport Problem

Prove Rich's Theorem



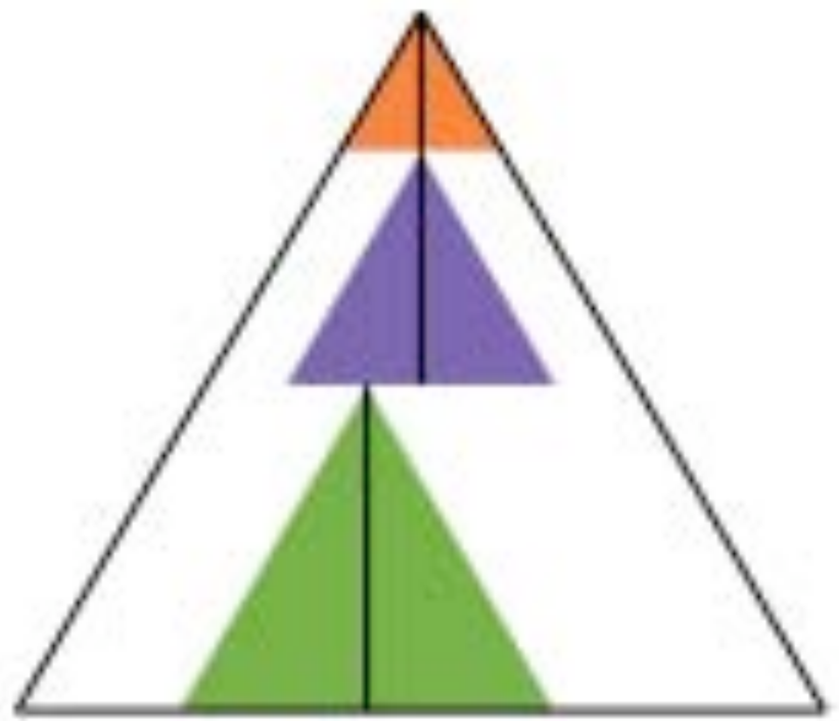
Proving the Airport Problem

Prove Rich's Theorem



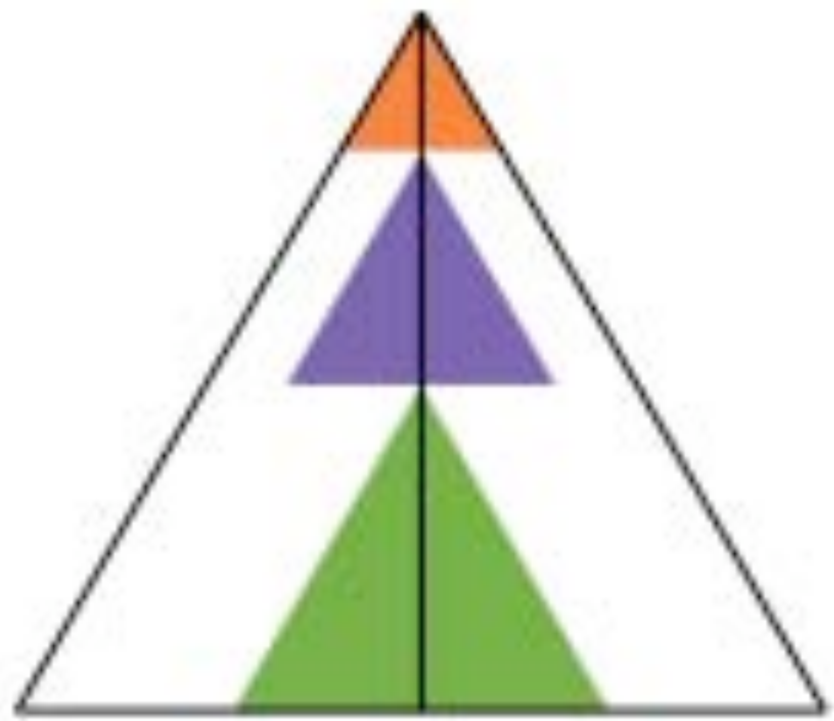
Proving the Airport Problem

Prove Rich's Theorem



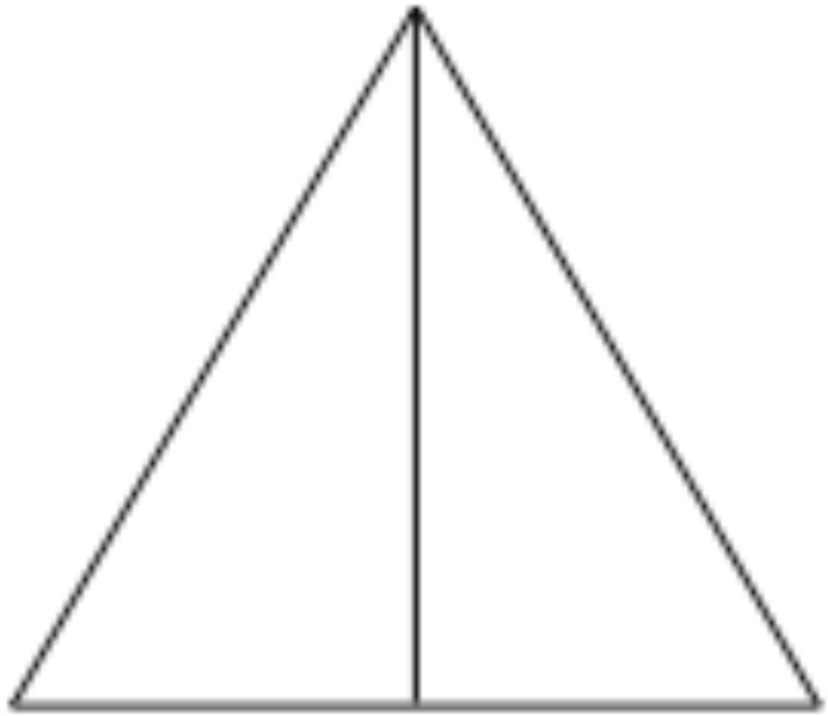
Proving the Airport Problem

Prove Rich’s Theorem



Proving the Airport Problem

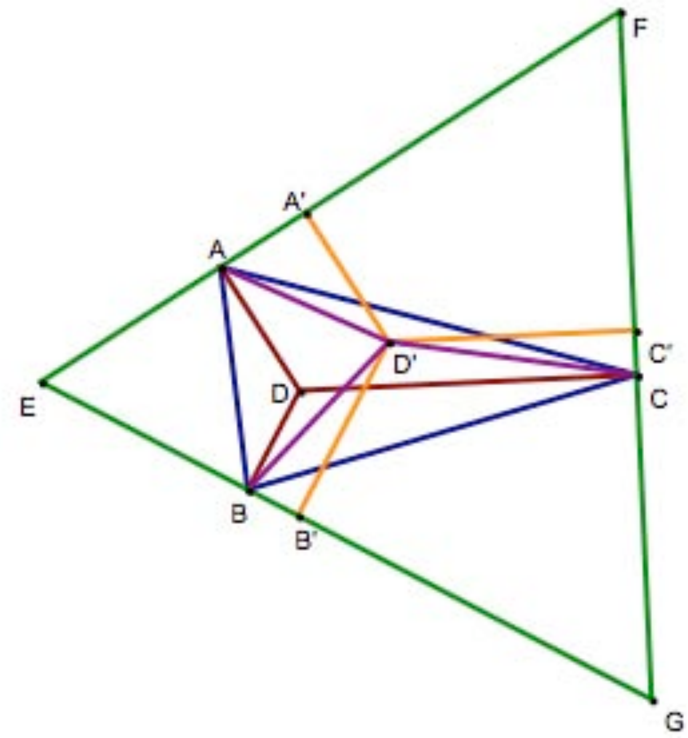
Prove Rich’s Theorem

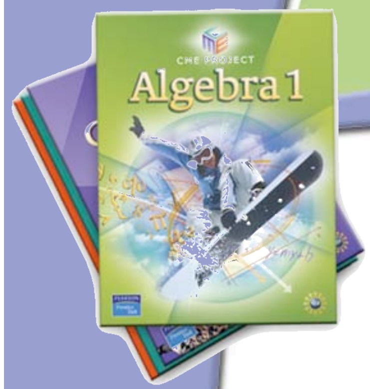


Proving the Airport Problem

Prove that the sum of the distances to *A*, *B*, *C* of any point *D* different from *F* is greater:

- By Rich's Theorem, AD + BD + CD = A'D' + B'D' + C'D'
- A'D' < AD', B'D' < BD', C'D' < CD'
- AD + BD + CD < AD' + BD' + CD'

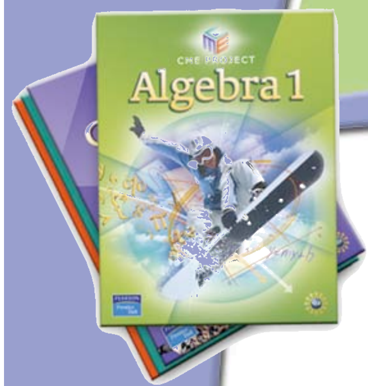




Experimenting

Find a Function that Fits This Table

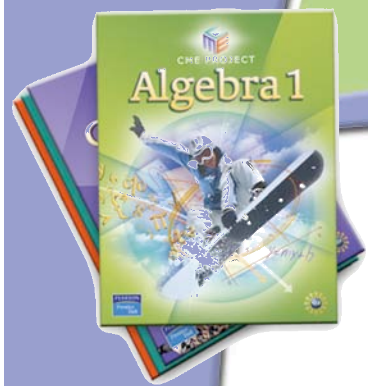
Input	Output
0	0
1	2
2	6
3	12
4	20



Experimenting

Find a Function that Fits This Table

Input	Output	Δ
0	0	2
1	2	4
2	6	6
3	12	8
4	20	



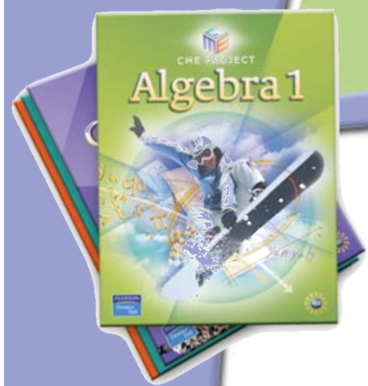
Defining Functions

Input	Output
0	0
1	2
2	6
3	12
4	20

$$f(n) = \begin{cases} 0, & n = 0 \\ f(n-1) + 2n, & n > 0 \end{cases}$$

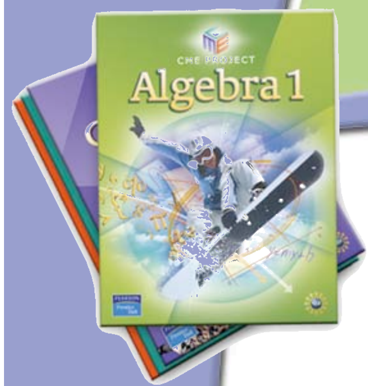
$$g(x) = x(x+1)$$

Will *f* and *g* be equal for every positive integer input?



Comparing Functions

$f(75)$	$= f(74) + 2 \cdot 75$	[definition of f]
	$= g(74) + 2 \cdot 75$	[CSS]
	$= 74 \cdot 75 + 2 \cdot 75$	[$g(74) = 74 \cdot 75$]
	$= 75 \cdot (74 + 2)$	[factor out 75]
	$= 75 \cdot 76$	[some arithmetic]
	$= g(75)$	[$g(75) = 75 \cdot 76$]



Comparing Functions

$$
\begin{aligned}
f(76) &= f(75) + 2 \cdot 76 && [\text{definition of } f] \\
&= g(75) + 2 \cdot 76 && [\text{just proved it}] \\
&= 75 \cdot 76 + 2 \cdot 76 && [g(75) = 75 \cdot 76] \\
&= 76 \cdot (75 + 2) && [\text{factor out } 76] \\
&= 76 \cdot 77 && [\text{some arithmetic}] \\
&= g(76) && [g(76) = 76 \cdot 77]
\end{aligned}
$$

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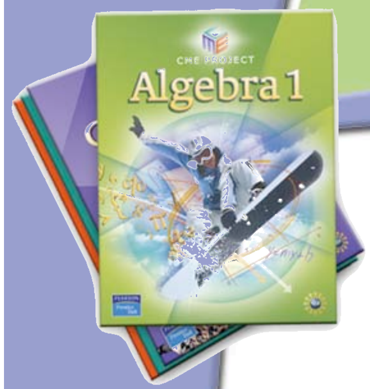
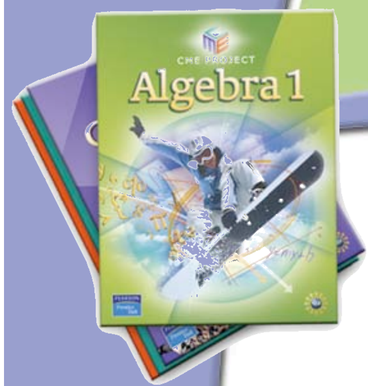
Comparing Functions

- Suppose a more powerful computer reported that $f(100) = g(100)$, but ran out of memory computing $f(101)$. Are f and g equal at 101?
- Imagine that a computer reported that $f(n–1) = g(n–1)$, but ran out of memory computing $f(n)$. Are f and g equal at n? How do you know?



Mathematical Induction

$$
\begin{aligned}
f(\mathrm{n}) &= f(n - 1) + 2n && \text{[definition of } f\text{]} \\
&= g(n - 1) + 2n && \text{[BICSS]} \\
&= (n - 1)\cdot n + 2n && [g(n-1) = (n-1)\cdot n] \\
&= n(n - 1 + 2) && \text{[factor out } n\text{]} \\
&= n(n + 1) && \text{[some arithmetic]} \\
&= g(n) && [g(n) = n(n + 1)]
\end{aligned}
$$

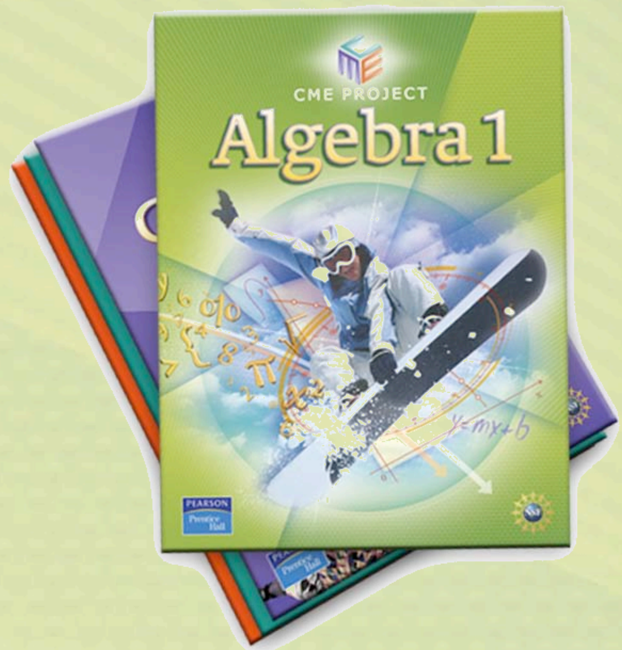


Mathematical Induction

- Students were very clear about what they are proving
- Students never felt they were “assuming what they want to prove”
- The limit of the calculator shows students that they cannot, in fact, check that the functions are equal for any input

CME Project Availability Dates

- **Algebra 1, Geometry, and Algebra 2**
  - **Available right now!**
- **Precalculus**
  - **Available Summer 2008**



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Developing Habits of Mind Workshops

- Explore mathematics content using *CME Project* materials
- Learn about pedagogical tools and style including mathematical representations, word problems, and skills practice
- Address issues of implementation, differentiation, and assessment
- Network with educators from across the country

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Location

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