REASONING AND SENSE MAKING WITH TECHNOLOGY
SOME EXAMPLES FROM ALGEBRA AND FUNCTIONS

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(special thanks to Kevin Waterman)

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OUTLINE

1. Introduction

2. Function Equality
   - Agreeing to Disagree
   - Equal Functions

3. Up a Notch
   - Resolving Recurrences
   - One for the Road
There are three uses of “this kind” of technology that can help students build ideas:

1. Reduce computational overhead
2. Construct and perform experiments
3. Build computational models of mathematical objects
Find a function that agrees with this table.

<table>
<thead>
<tr>
<th>INPUT</th>
<th>OUTPUT</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
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<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>
What would you do if . . .

<table>
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</table>

Sasha says, “I know, I know, it’s

\[ f(n) = n^5 - 10n^4 + 35n^3 - 50n^2 + 26n + 1, \]"
Variation 1

Find some polynomial functions that agree with this table.

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</tr>
</tbody>
</table>
Suppose you have two functions, \( f \) and \( g \) that agree on \( \{0, 1, 2, 3, 4\} \).

If \( f \) and \( g \) are polynomial functions, then \( g - f \) is a polynomial function with zeros at \( \{0, 1, 2, 3, 4\} \).

By the factor theorem, \( g - f \) has as factors \( x, x - 1, x - 2, x - 3, x - 4 \).

Hence
\[
(g - f)(x) = \text{something} \cdot x(x - 1)(x - 2)(x - 3)(x - 4)
\]

and
\[
g(x) = f(x) + k \cdot x(x - 1)(x - 2)(x - 3)(x - 4)
\]
**VARIATION 2**

Find a function that agrees with this table.

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<td>4</td>
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</tr>
</tbody>
</table>
Now we have two models:

\[
f(n) = 2n + 1 \quad \quad g(n) = \begin{cases} 
1 & n = 0 \\
g(n - 1) + 2 & n > 0 
\end{cases}
\]

\[f \equiv g\]
TWO MODELS

\[ f(n) = 2n + 1 \quad g(n) = \begin{cases} 
1 & n = 0 \\
g(n - 1) + 2 & n > 0 
\end{cases} \]

Suppose on your handheld, \( f(n) = g(n) \) for \( 0 \leq n \leq 64 \), but \( f(65) \) reports 131 and \( g(65) \) reports an error.

\[
g(65) = g(64) + 2 \quad \text{(this is how } g \text{ is defined)} \\
= f(64) + 2 \quad \text{(CSS)} \\
= (2 \cdot 64 + 1) + 2 \quad \text{(this is how } f \text{ is defined)} \\
= (2 \cdot 64 + 2) + 1 \quad \text{(arithmetic)} \\
= (2 \cdot 65) + 1 \quad \text{(more arithmetic)} \\
= f(65) \quad \text{(this is how } f \text{ is defined)}
\]


**TWO MODELS**

\[
\begin{align*}
  f(n) &= 2n + 1 \\
  g(n) &= \begin{cases} 
  1 & n = 0 \\
  g(n-1) + 2 & n > 0 
\end{cases}
\end{align*}
\]

Suppose on your handheld, \( f(n) = g(n) \) for \( 0 \leq n \leq 254 \), but \( f(255) \) reports 510 and \( g(255) \) reports an error.

\[
\begin{align*}
  g(255) &= g(254) + 2 \quad \text{(this is how } g \text{ is defined)} \\
  &= f(254) + 2 \quad \text{(CSS)} \\
  &= (2 \cdot 254 + 1) + 2 \quad \text{(this is how } f \text{ is defined)} \\
  &= (2 \cdot 254 + 2) + 1 \quad \text{(arithmetic)} \\
  &= (2 \cdot 255) + 1 \quad \text{(more arithmetic)} \\
  &= f(255) \quad \text{(this is how } f \text{ is defined)}
\end{align*}
\]
**TWO MODELS**

\[ f(n) = 2n + 1 \]
\[ g(n) = \begin{cases} 
1 & n = 0 \\
g(n-1) + 2 & n > 0
\end{cases} \]

Suppose on your (virtual) handheld, \( f(n) = g(n) \) for \( 0 \leq n \leq k - 1 \), but \( f(k) \) reports \( 2k + 1 \) and \( g(k) \) reports an error.

\[
g(k) = g(k - 1) + 2 \quad \text{(this is how } g \text{ is defined)}
\]
\[
= f(k - 1) + 2 \quad \text{(VCSS)}
\]
\[
= (2 \cdot (k - 1) + 1) + 2 \quad \text{(this is how } f \text{ is defined)}
\]
\[
= (2 \cdot (k - 1) + 2) + 1 \quad \text{(arithmetic)}
\]
\[
= (2 \cdot k) + 1 \quad \text{(algebra)}
\]
\[
= f(k) \quad \text{(this is how } f \text{ is defined)}
\]
Try Some

- Pick your favorite table from the first page of the handout.  
  - Don’t pick 1a.

- Find a closed form and a recursive model that agrees with your table.

- Are your two models equal on $\mathbb{Z}^\geq 0$?  
  - If not, find a place where they disagree.  
  - If so, prove it.
Experiment with this puppy:

\[ g(n) = \begin{cases} 
2 & \text{if } n = 0 \\
2 & \text{if } n = 1 \\
2g(n - 1) + 3g(n - 2) & \text{if } n > 1 
\end{cases} \]
A mixed methods

How about this one?

\[ h(n) = \begin{cases} 
2 & n = 0 \\
3h(n - 1) - 2 & n > 1 
\end{cases} \]
You want to buy a car that costs $25000, and you can put $1000 down. The annual interest rate is 5%. What monthly payment would let you own the car after 48 months?

*Hint*: What you owe at the end of a month is what you owed at the start of the month, multiplied by $1 + \frac{0.05}{12}$, minus the monthly payment.

- Write a function $b(n, m)$ that gives the balance on the loan at the end of $n$ months, with a monthly payment of $m$.
- Then find the $m$ that makes $b(48, m) = 0$.
- Is there a closed form for $b$?
THANKS

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