Algebra in the Age of CAS: Implications for the High School Curriculum

Examples from *The CME Project*

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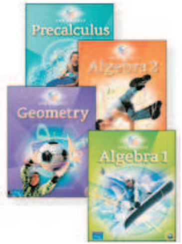
CSMC, 2008

Outline

THE CME PROJECT

- An NSF-funded coherent 4-year curriculum
- Published by Pearson
- Follows the traditional American course structure
- Uses the TI-Nspire in all 4 years
- Makes essential use of a CAS in the last two years
- Organized around mathematical habits of mind



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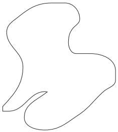
  - Is there a line that cuts the area of  in half?

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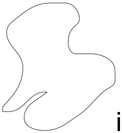
  - Is there a line that cuts the area of  in half?
  - Is the average of two averages the average of the lot?

Algebraic Habits of Mind

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ALGEBRAIC HABITS OF MIND

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- Seeking structural similarity:
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- Reasoning about operations
  "If the rules of arithmetic are to hold, $5^{\frac{2}{3}}$ must be a number whose cube is 5^2."

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- allow students to build computational models of algebraic objects that have no faithful physical counterparts, highlighting similarities in algebraic structure.

Experimenting: A weird function

The number of factors over $\mathbb{Z}$ of $x^n - 1$ as a function of n.

n	number of factors of $x^n - 1$
1	
2	
3	
4	
5	
6	
7	
8	
9	

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Conjectures? . . .

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- There are always at least two factors:

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$$\vdots$$

- A general conjecture gradually emerges

Reducing overhead: The Polynomial Factor Game

The CMP version:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30

Reducing overhead: The Polynomial Factor Game

The CME version:

$x-1$	x^2-1	x^3-1	x^4-1	x^5-1
x^6-1	x^7-1	x^8-1	x^9-1	$x^{10}-1$
$x^{11}-1$	$x^{12}-1$	$x^{13}-1$	$x^{14}-1$	$x^{15}-1$
$x^{16}-1$	$x^{17}-1$	$x^{18}-1$	$x^{19}-1$	$x^{20}-1$
$x^{21}-1$	$x^{22}-1$	$x^{23}-1$	$x^{24}-1$	$x^{25}-1$
$x^{26}-1$	$x^{27}-1$	$x^{28}-1$	$x^{29}-1$	$x^{30}-1$

Scratchpad

Conjectures?

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$$
\begin{aligned}
x^{12} - 1 &= \left(x^3\right)^4 - 1 \\
&= (\clubsuit)^4 - 1 \\
&= (\clubsuit - 1)\left(\clubsuit^3 + \clubsuit^2 + \clubsuit + 1\right) \\
&= \left(x^3 - 1\right)\left((x^3)^3 + (x^3)^2 + (x^3) + 1\right) \\
&= \left(x^3 - 1\right)\left(x^9 + x^6 + x^3 + 1\right)
\end{aligned}
$$

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  This is much harder, and it requires some facility with De Moivre's theorem and with *roots of unity*: complex numbers that are the roots of the equation

$$x^n - 1 = 0$$

Modeling: Roots of Unity

De Moivre's Theorem implies

- The roots of $x^n - 1 = 0$ are

$$\left\{ \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \quad \middle| \quad 0 \le k < n \right\}$$

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$$1, \zeta, \zeta^2, \zeta^3, \ldots, \zeta^{n-1}$$

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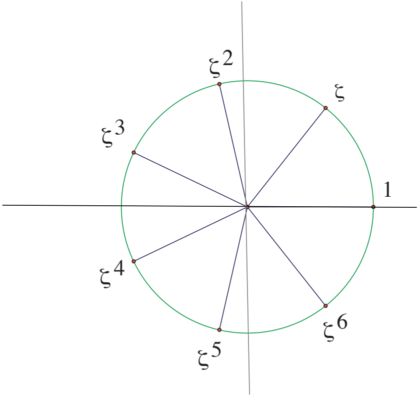
- These roots lie on the vertices of a regular n-gon of radius 1 in the complex plane

Examples

Modeling: Roots of Unity

Here are the 7th roots of unity.



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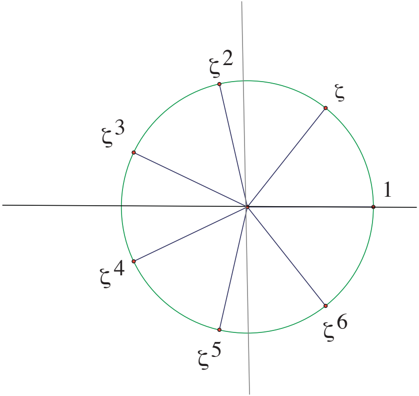

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- So $(\zeta + \zeta^6)$, $(\zeta^2 + \zeta^5)$, and $(\zeta^3 + \zeta^4)$ are real numbers.

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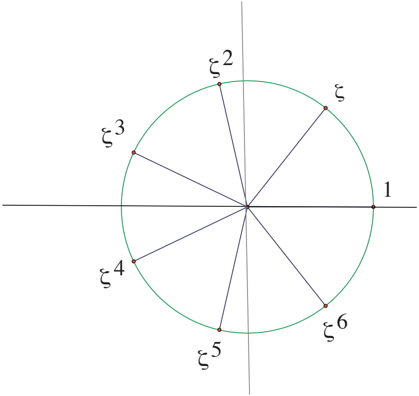
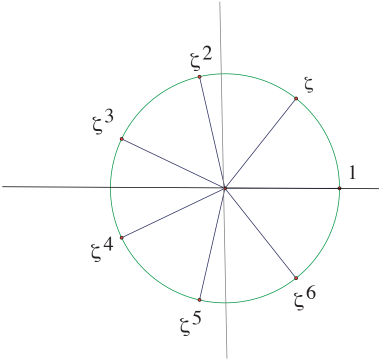


- The six non-real roots come in conjugate pairs.
- So $(\zeta + \zeta^6)$, $(\zeta^2 + \zeta^5)$, and $(\zeta^3 + \zeta^4)$ are real numbers.
- What cubic equation over $\mathbb{R}$ has these three numbers as roots?

Modeling: Roots of Unity



Let

$$\alpha = \zeta + \zeta^6$$
$$\beta = \zeta^2 + \zeta^5$$
$$\gamma = \zeta^3 + \zeta^4$$

To find an equation satisfied by α, β, and γ, we need to find

- $\alpha + \beta + \gamma$
- $\alpha\beta + \alpha\gamma + \beta\gamma$
- $\alpha\beta\gamma$

One at a time. . .

Modeling: Roots of Unity

The Sum:

Since $\alpha = \zeta + \zeta^6$, $\beta = \zeta^2 + \zeta^5$, and $\gamma = \zeta^3 + \zeta^4$, we have

$$\alpha + \beta + \gamma = \zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta$$

But

$$x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$$

So,

$$\zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta = -1$$

Modeling: Roots of Unity

The Product:

$$\alpha\beta\gamma = \left(\zeta + \zeta^6\right)\left(\zeta^2 + \zeta^5\right)\left(\zeta^3 + \zeta^4\right)$$

We can get the form of the expansion by expanding

$$\left(x + x^6\right)\left(x^2 + x^5\right)\left(x^3 + x^4\right)$$

Time for a CAS

Modeling: Roots of Unity

So,

$$\left(x + x^6\right)\left(x^2 + x^5\right)\left(x^3 + x^4\right) =$$

$$x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^9 + x^7 + x^6$$

But if we replace x by ζ, we can replace x^7 by 1...

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But if we replace x by ζ, we can replace x^7 by $1\ldots$
So, if the above expression is written as

$$(x^7 - 1)q(x) + r(x)$$

then replacing x by ζ will produce $r(\zeta)$

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So $\alpha\beta\gamma = 1$

Modeling: Roots of Unity

What about the “beast”? Well, $\alpha\beta + \alpha\gamma + \beta\gamma =$

$$\left(\zeta + \zeta^6\right)\left(\zeta^2 + \zeta^5\right) +$$
$$\left(\zeta + \zeta^6\right)\left(\zeta^3 + \zeta^4\right) +$$
$$\left(\zeta^2 + \zeta^5\right)\left(\zeta^3 + \zeta^4\right)$$

Time for a CAS

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Time for a CAS

So $\alpha\beta + \alpha\gamma + \beta\gamma = -2$ and our cubic is

$$x^3 + x^2 - 2x - 1 = 0$$

Modeling: Roots of Unity

- In this informal way, students preview the idea that one can model $\mathbb{Q}(\zeta)$ by "remainder arithmetic" in $\mathbb{Q}(x)$, using $x^7 - 1$ as a divisor.

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- This previews Kronecker's construction of splitting fields for algebraic equations.

USING A CAS IN HIGH SCHOOL: OTHER EXAMPLES

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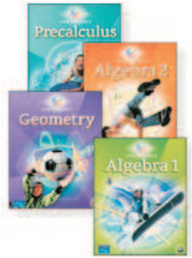
1. Experiment with algebra: Chebyshev polynomials
2. Reduce computational overhead: Lagrange interpolation and Newton's Difference Formula
3. Use polynomials as modeling tools: Generating functions

For More Information

- For the program:
  - `www.pearsonschool.org/cme`
  - Al Cuoco (`acuoco@edc.org`)
- For summer workshops
  - `www.edc.org/cmeproject`
  - Melody Hachey (`mhachey@edc.org`)
  - Sarah Sword (`ssword@edc.org`)

Availability



- Algebra 1, Geometry, and Algebra 2
  - **Now**
- Precalculus
  - **This Summer**

Workshops this summer: August 4–8.