Some Organizing Principles for High School Algebra

An Example: *The CME Project*

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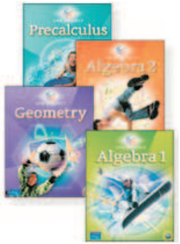
OUTLINE

1. SOME BACKGROUND
   - What is *The CME Project*?
   - Some goals of the program
   - The *Habits of Mind* approach
2. SOME ALGEBRAIC HABITS OF MIND
   - Examples of habits indigenous to algebra
   - How they play out in high school algebra
3. CONCLUSION

The CME Project

- An NSF-funded coherent 4-year curriculum
- Published by Pearson
- Follows the traditional American course structure
- Uses the TI-Nspire in all 4 years
- Makes essential use of a CAS in the last two years
- Organized around mathematical habits of mind



Some Goals of *The CME Project*

1. Make connections with algebra as a discipline

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Addressing item 3 helps one address the first two.

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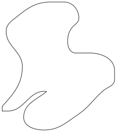
    - Is there a line that cuts the area of  in half?

THE *Habits of Mind* APPROACH

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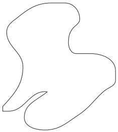
  - Is there a line that cuts the area of  in half?
  - Is the average of two averages the average of the lot?

ALGEBRAIC HABITS OF MIND

- Seeking regularity in repeated calculations
- "Chunking" (changing variables in order to hide complexity)
- Reasoning about and picturing calculations and operations
- Purposefully transforming and interpreting expressions
- Seeking and modeling structural similarities

Example 1(a): Graphing

Graph

$$16\,x^2 - 96\,x + 25\,y^2 - 100\,y - 156 = 0$$

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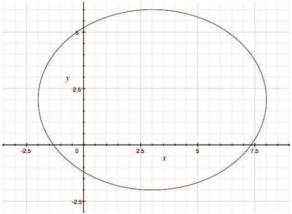
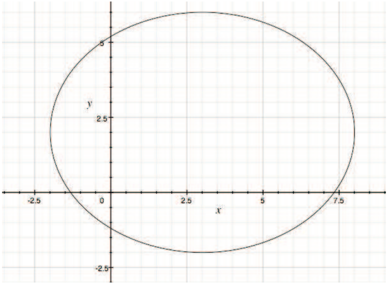
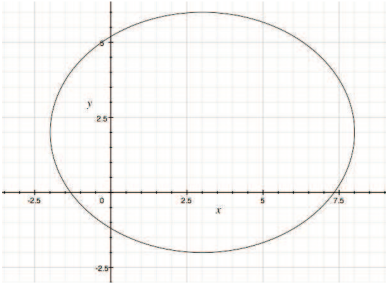


EXAMPLE 1(A): GRAPHING . . .



Is $(7.5, 3.75)$ on the graph?

Example 1(a): Graphing . . .



Is $(7.5, 3.75)$ on the graph?

This led to the idea that “equations are point testers.”

Example 1(b): Equations of lines

Why is "linearity" so hard for students?

EXAMPLE 1(B): EQUATIONS OF LINES

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- Slope is defined initially *between two points* : $m(A, B)$

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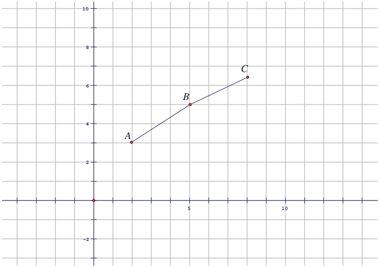
Why is “linearity” so hard for students?

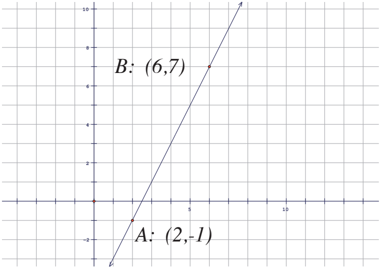
- Slope is defined initially *between two points* : $m(A, B)$

Basic assumption: A, B, and C are collinear $\Leftrightarrow m(A, B) = m(B, C)$



Example 1(b): Equations of lines

What is the equation of $\ell = \overleftrightarrow{AB}$ if $A = (2, -1)$ and $B = (6, 7)$?



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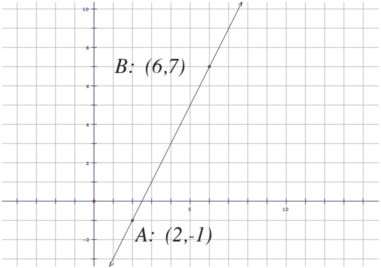
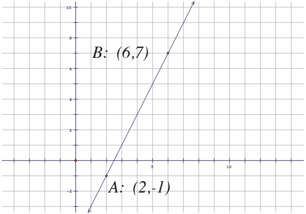


Try some points, keeping track of the steps...

EXAMPLE 1(B): EQUATIONS OF LINES

- $A = (2, -1)$ and $B = (6, 7)$
- $m(A, B) = 2$



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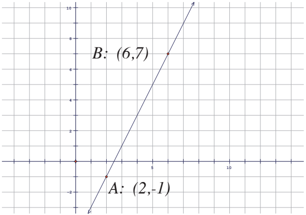

- Test $C = (3, 4)$:
  $m(B, C) = \frac{4-7}{3-6} \stackrel{?}{=} 2 \Rightarrow$ Nope

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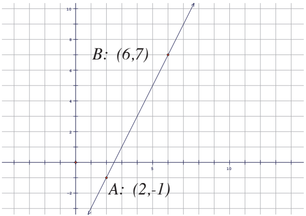

- Test $C = (3, 4)$: $m(B, C) = \frac{4-7}{3-6} \stackrel{?}{=} 2 \Rightarrow$ Nope
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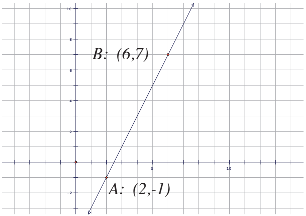

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- Test $C = (x, y)$: $m(B, C) = \frac{y-7}{x-6} \stackrel{?}{=} 2$

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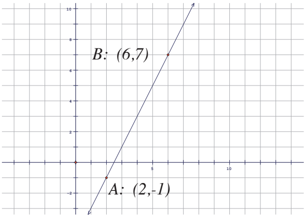

- Test $C = (3, 4)$:
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- Test $C = (x, y)$:
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And the equation is $\frac{y-7}{x-6} = 2$

Other applications

This habit of seeking regularity in repeated calculations is useful in other situations

- finding equations that model word problems (Roskam)
- finding equations for curves
- finding functions that agree with tables
- establishing algebraic identities
- establishing proofs by mathematical induction

Example 2: Factoring

Monic quadratics:

"Sum-Product" problems

$$x^2 + 14x + 48$$

Find two numbers whose sum is 14 and whose product is 48.

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Find two numbers whose sum is 14 and whose product is 48.

$$(x + 6)(x + 8)$$

FACTORING

What about this one?

$$49x^2 + 35x + 6$$

Factoring

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$$49x^2 + 35x + 6 = (7x)^2 + 5(7x) + 6$$

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$$= (\clubsuit + 3)(\clubsuit + 2)$$

Factoring

What about this one?

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$$\begin{aligned} 49x^2 + 35x + 6 &= (7x)^2 + 5(7x) + 6 \\ &= \clubsuit^2 + 5\clubsuit + 6 \\ &= (\clubsuit + 3)(\clubsuit + 2) \\ &= (7x + 3)(7x + 2) \end{aligned}$$

FACTORING

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&= z^2 + 31z + 210 \\
&= (z + 21)(z + 10) \\
&= (6x + 21)(6x + 10) \\
&= 3(2x + 7)\,2(3x + 5) \\
&= \cancel{6}(2x + 7)(3x + 5)
\end{aligned}$$

Other Applications

This habit of “chunking” is useful in other situations

- normalizing higher degree polynomials
- deriving Cardano’s formula
- solving trig and exponential equations
- completing the square
- analyzing affine transformations of graphs

Example 3: Picturing Calculations

- $(x-1)(x^4+x^3+x^2+x+1)$

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- $(x-1)(x^4+x^3+x^2+x+1)$
- $(a+b)^2-(a-b)^2$
- $3(x-1)(x-3)+5(x-1)(x-2)-7(x-2)(x-3)$

Other applications

This habit of picturing calculations is useful in many situations

The best theorems of mathematics have the form, "If you do calculation X, you will find the result Y." To understand the meaning of such a theorem, you must be able to imagine the calculation, but the actual calculation is rarely needed; if it is *needed, you need to be able to imagine alternative ways of doing it in order to select a good one.*

—Harold Edwards

Example 4(a): Transforming and Interpreting

Let $f(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^3$

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$$
\begin{aligned}
&(x + x^2 + x^3 + x^4 + x^5 + x^6)\times \\
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&(x + x^2 + x^3 + x^4 + x^5 + x^6)\dots
\end{aligned}
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$$\begin{array}{c}(x + x^2 + x^3 + x^4 + x^5 + x^6)\times \\ (x + x^2 + x^3 + x^4 + x^5 + x^6)\times \\ (x + x^2 + x^3 + x^4 + x^5 + x^6)\ldots\end{array}$$

- What is the most likely sum when 3 dice are tossed?

  Experiment

Define $f(x)=(x+x^2+x^3+x^4+x^5+x^6)^3$	*Done*
$\text{expand}(f(x))$	
$x^{18}+3\cdot x^{17}+6\cdot x^{16}+10\cdot x^{15}+15\cdot x^{14}+21\cdot x^{13}+25\cdot x^{12}+27\cdot x^{11}+27\cdot x^{10}+25\cdot x^{9}+21\cdot x^{8}+15\cdot x^{7}+10\cdot x^{6}+6\cdot x^{5}+3\cdot x^{4}+x^{3}$	
$f(^{-}1)$	0

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$$
\begin{gathered}
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(x + x^2 + x^3 + x^4 + x^5 + x^6)\times \\
(x + x^2 + x^3 + x^4 + x^5 + x^6)\dots
\end{gathered}
$$

- What is the most likely sum when 3 dice are tossed?

  Experiment

- What does the value of $f(-1)$ say about the distribution of sums?

Example 4(b): Transforming and Interpreting

Heron's formula for the area of a triangle with side-lengths a, b, and c is

$$4 \cdot h(a,b,c) = \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

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Give geometric and algebraic explanations for your answers:

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- When is $h(a, b, c) = 0$?

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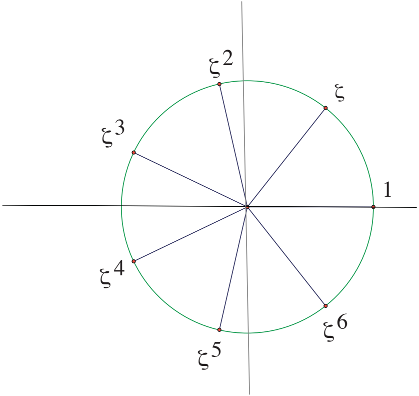
- Why is $h(a,b,c) = h(c,a,b) = h(b,c,a)$?
- When is $h(a,b,c) = 0$?
- Express $h(3a,3b,3c)$ in terms of $h(a,b,c)$

Example 5: Modeling with Polynomials

Here are the 7th roots of unity.



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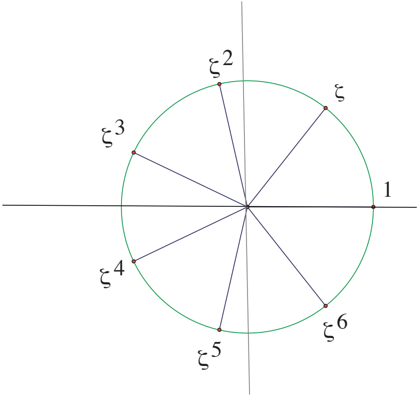

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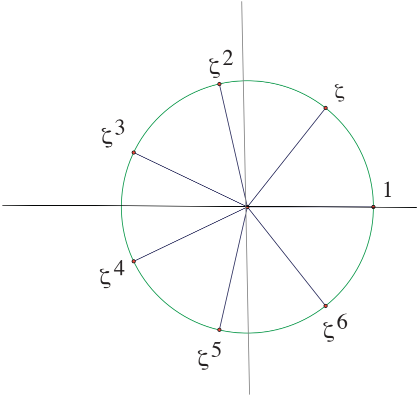

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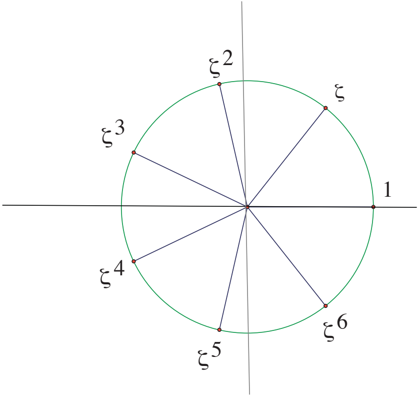

- ζ and its integer powers are roots of $x^7 - 1 = 0$.
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- So $(\zeta + \zeta^6)$, $(\zeta^2 + \zeta^5)$, and $(\zeta^3 + \zeta^4)$ are real numbers.

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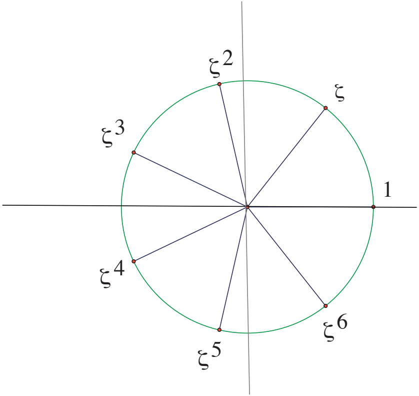
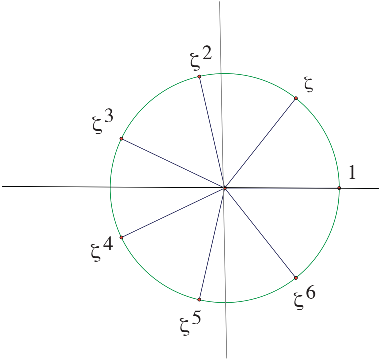


- ζ and its integer powers are roots of $x^7 - 1 = 0$.
- The six non-real roots come in conjugate pairs.
- So $(\zeta + \zeta^6)$, $(\zeta^2 + \zeta^5)$, and $(\zeta^3 + \zeta^4)$ are real numbers.
- What cubic equation over $\mathbb{R}$ has these three numbers as roots?

Example 5: Modeling with Polynomials



Let

$$\alpha = \zeta + \zeta^6$$
$$\beta = \zeta^2 + \zeta^5$$
$$\gamma = \zeta^3 + \zeta^4$$

To find an equation satisfied by α, β, and γ, we need to find

- $\alpha + \beta + \gamma$
- $\alpha\beta + \alpha\gamma + \beta\gamma$
- $\alpha\beta\gamma$

One at a time. . .

Example 5: Modeling with Polynomials

The Sum:

Since $\alpha = \zeta + \zeta^6$, $\beta = \zeta^2 + \zeta^5$, and $\gamma = \zeta^3 + \zeta^4$, we have

$$\alpha + \beta + \gamma = \zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta$$

But

$$x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$$

So,

$$\zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta = -1$$

Example 5: Modeling with Polynomials

The Product:

$$\alpha\beta\gamma = \left(\zeta + \zeta^6\right)\left(\zeta^2 + \zeta^5\right)\left(\zeta^3 + \zeta^4\right)$$

We can get the form of the expansion by expanding

$$\left(x + x^6\right)\left(x^2 + x^5\right)\left(x^3 + x^4\right)$$

CAS

$$\text{expand}\left((x+x^6)\cdot(x^2+x^5)\cdot(x^3+x^4)\right) \quad x^{15}+x^{14}+x^{12}+x^{11}+x^{10}+x^9+x^7+x^6$$

EXAMPLE 5: MODELING WITH POLYNOMIALS

So,

$$\left(x + x^6\right)\left(x^2 + x^5\right)\left(x^3 + x^4\right) =$$

$$x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^9 + x^7 + x^6$$

But if we replace x by ζ, we can replace x^7 by 1...

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But if we replace x by ζ, we can replace x^7 by 1...
So, if the above expression is written as

$$(x^7-1)q(x)+r(x)$$

then replacing x by ζ will produce $r(\zeta)$

CAS

$\text{polyRemainder}\left(\left(x+x^{6}\right)\cdot\left(x^{2}+x^{5}\right)\cdot\left(x^{3}+x^{4}\right),x^{7}-1\right)$ $x^{6}+x^{5}+x^{4}+x^{3}+x^{2}+x+2$

Example 5: Modeling with Polynomials

So,

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$$x^{15}+x^{14}+x^{12}+x^{11}+x^{10}+x^9+x^7+x^6$$

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So, if the above expression is written as

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CAS

So $\alpha\beta\gamma=1$

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What about the "beast"? Well, $\alpha\beta + \alpha\gamma + \beta\gamma =$

$$\left(\zeta + \zeta^6\right)\left(\zeta^2 + \zeta^5\right) + \\ \left(\zeta + \zeta^6\right)\left(\zeta^3 + \zeta^4\right) + \\ \left(\zeta^2 + \zeta^5\right)\left(\zeta^3 + \zeta^4\right)$$

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So $\alpha\beta + \alpha\gamma + \beta\gamma = -2$ and our cubic is

$$x^3 + x^2 - 2x - 1 = 0$$

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- This previews Kronecker's construction of splitting fields for algebraic equations.

Other applications

This habit of seeking and modeling structural similarities in algebraic systems is useful in other situations

- Matrices $\leftrightarrow$ linear transformations of the plane
- Arithmetic with integers $\leftrightarrow$ arithmetic with polynomials
- $\mathbb{C} \leftrightarrow \mathbb{R}[i]$
- Trig identities via arithmetic with complex numbers
- $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \leftrightarrow a + bi$
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow \frac{ax+b}{cx+d}$

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An organization around algebraic habits of mind

- helps students see some coherence in algebra
- provides students with general-purpose mathematical approaches
- helps align school algebra with algebra as a scientific discipline
- helps students develop habits that are genuinely useful in the world
- is one of many valid ways to develop school algebra

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- Looking extreme cases
- Passing to the limit
- Extension by continuity
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These ways of thinking form a solid basis for courses in geometry and "precalculus."

THANKS

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