Algebra in the Age of CAS
Examples from *The CME Project* (and beyond)

A story about CAS and Theorems

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Center for Mathematics Education
EDC

USACAS, 2008

CME PROJECT

See, this is easy for you to say, and easy for me to understand, but it's not code for a CAS.

—Cleve Moler

OUTLINE

CME PROJECT

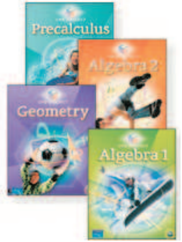
JUST FOR FUN

Input	Output	Δ	Δ^2	Δ^3
0	1	-2	14	12
1	-1	12	26	12
2	11	38	38	12
3	49	76	50	12
4	125	126	62	12
5	251	188	74	
6	439	262		
7	701			

scratchpad

THE CME PROJECT

- An NSF-funded coherent 4-year curriculum
- Published by Pearson
- Follows the traditional American course structure
- Uses the TI-Nspire in all 4 years
- Makes essential use of a CAS in the last two years
- Organized around mathematical habits of mind



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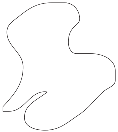
    - Is there a line that cuts the area of  in half?

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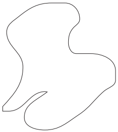
    - Is there a line that cuts the area of  in half?
    - Is the average of two averages the average of the lot?

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Algebraic Habits of Mind

- Seeking regularity in repeated calculations.
- “Chunking” (changing variables to hide complexity).
- Reasoning about and picturing calculations.
- Purposefully transforming and interpreting expressions to reveal hidden meaning.
- Seeking and modeling structural similarities in algebraic systems.
- Reasoning about and extending operations.

Example: Reasoning about and picturing calculations

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 \stackrel{?}{=}$$

CME PROJECT

EXAMPLE: REASONING ABOUT AND PICTURING CALCULATIONS

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CME PROJECT

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$$x^5 - 1$$

CME PROJECT

Example: Chunking

Factor

$$6x^2 + 31x + 35$$

EXAMPLE: CHUNKING

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$$6(6x^2 + 31x + 35) = (6x)^2 + 31(6x) + 210$$

CME PROJECT

EXAMPLE: CHUNKING

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Example: Chunking

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$$\begin{aligned} 6(6x^2 + 31x + 35) &= (6x)^2 + 31(6x) + 210 \\ &= \clubsuit^2 + 31\clubsuit + 210 \\ &= (\clubsuit + 21)(\clubsuit + 10) \end{aligned}$$

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CME PROJECT

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$$\not{6}(6x^2 + 31x + 35) = \not{6}(2x + 7)(3x + 5)$$

CME PROJECT

Our uses of CAS

CAS environments. . .

- provide students a platform for experimenting with algebraic expressions and other mathematical objects in the same way that calculators can be used to experiment with numbers.

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CAS environments. . .

- provide students a platform for experimenting with algebraic expressions and other mathematical objects in the same way that calculators can be used to experiment with numbers.
- make tractable and to enhance many beautiful classical topics, historically considered too technical for high school students, by reducing computational overhead.
- allow students to build computational models of algebraic objects that have no faithful physical counterparts, highlighting similarities in algebraic structure.

Experimenting: A weird function

The number of factors over $\mathbb{Z}$ of $x^n - 1$ as a function of n.

n	number of factors of $x^n - 1$
1	
2	
3	
4	
5	
6	
7	
8	
9	

CME PROJECT

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CME PROJECT

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CME PROJECT

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4	2
5	2
6	4
7	2
8	4
9	3

Conjectures? . . .

CME PROJECT

Experimenting: A weird function

Things that have come up in class:

- There are always at least two factors:

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1)$$

CME PROJECT

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$$\vdots$$

- A general conjecture gradually emerges

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EXPERIMENTING: A WEIRD FUNCTION

A closer look at the factors:

1	$-1+x$
2	$(-1+x)(1+x)$
3	$(-1+x)(1+x+x^2)$
4	$(-1+x)(1+x)(1+x^2)$
5	$(-1+x)(1+x+x^2+x^3+x^4)$
6	$(-1+x)(1+x)(1-x+x^2)(1+x+x^2)$
7	$(-1+x)(1+x+x^2+x^3+x^4+x^5+x^6)$
8	$(-1+x)(1+x)(1+x^2)(1+x^4)$
9	$(-1+x)(1+x+x^2)(1+x^3+x^6)$
10	$(-1+x)(1+x)(1-x+x^2-x^3+x^4)(1+x+x^2+x^3+x^4)$

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Experimenting: A weird function

The degrees of the factors:

1	1		11	1, 10
2	1, 1		12	1, 1, 2, 2, 2, 4
3	1, 2		13	1, 12
4	1, 1, 2		14	1, 1, 6, 6
5	1, 4		15	1, 2, 4, 8
6	1, 1, 2, 2		16	1, 1, 2, 4, 8
7	1, 6		17	1, 16
8	1, 1, 2, 4		18	1, 1, 2, 2, 6, 6
9	1, 2, 6		19	1, 18
10	1, 1, 4, 4		20	1, 1, 2, 4, 4, 8

Scratchpad

Experimenting: A weird function

Questions, Conjectures, Ideas. . .

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Experimenting: A weird function

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Experimenting: A weird function

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  - When are the factors of $x^m - 1$ among those of $x^n - 1$?

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To set the stage, we need an interlude from *The CME Project* Precalculus. . .

CME PROJECT

Reducing overhead: The Polynomial Factor Game

The CMP version:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30

CME PROJECT

REDUCING OVERHEAD: THE POLYNOMIAL FACTOR GAME

The CME version:

$x - 1$	$x^2 - 1$	$x^3 - 1$	$x^4 - 1$	$x^5 - 1$
$x^6 - 1$	$x^7 - 1$	$x^8 - 1$	$x^9 - 1$	$x^{10} - 1$
$x^{11} - 1$	$x^{12} - 1$	$x^{13} - 1$	$x^{14} - 1$	$x^{15} - 1$
$x^{16} - 1$	$x^{17} - 1$	$x^{18} - 1$	$x^{19} - 1$	$x^{20} - 1$
$x^{21} - 1$	$x^{22} - 1$	$x^{23} - 1$	$x^{24} - 1$	$x^{25} - 1$
$x^{26} - 1$	$x^{27} - 1$	$x^{28} - 1$	$x^{29} - 1$	$x^{30} - 1$

Scratchpad

Conjectures?

CME PROJECT

Reducing overhead: The Polynomial Factor Game

Things that come up:

- “It’s the same as the middle school factor game.”

Reducing overhead: The Polynomial Factor Game

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CME PROJECT

Reducing overhead: The Polynomial Factor Game

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$$\begin{aligned} x^{12} - 1 &= \left(x^3\right)^4 - 1 \\ &= (\clubsuit)^4 - 1 \\ &= (\clubsuit - 1)\left(\clubsuit^3 + \clubsuit^2 + \clubsuit + 1\right) \\ &= \left(x^3 - 1\right)\left((x^3)^3 + (x^3)^2 + (x^3) + 1\right) \\ &= \left(x^3 - 1\right)\left(x^9 + x^6 + x^3 + 1\right) \end{aligned}$$

Reducing overhead: The Polynomial Factor Game

- If $x^m - 1$ is a factor of $x^n - 1$, m is a factor of n

Reducing overhead: The Polynomial Factor Game

- If $x^m - 1$ is a factor of $x^n - 1$, m is a factor of n

  This is much harder. One way to see it is to use De Moivre's theorem and *roots of unity*: complex numbers that are the roots of the equation

$$x^n - 1 = 0$$

Modeling: Roots of Unity

De Moivre's Theorem implies

- The roots of $x^n - 1 = 0$ are

$$\left\{\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \quad \middle| \quad 0 \leq k < n\right\}$$

Modeling: Roots of Unity

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- If $\zeta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, these roots are

$$1, \zeta, \zeta^2, \zeta^3, \ldots, \zeta^{n-1}$$

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$$1, \zeta, \zeta^2, \zeta^3, \ldots, \zeta^{n-1}$$

- These roots lie on the vertices of a regular n-gon of radius 1 in the complex plane

Examples

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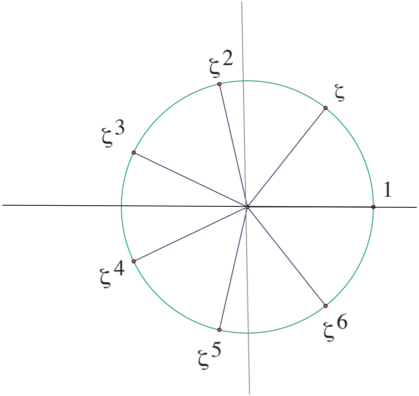
Modeling: Roots of Unity

Here are the 7th roots of unity.



- The six non-real roots come in conjugate pairs.

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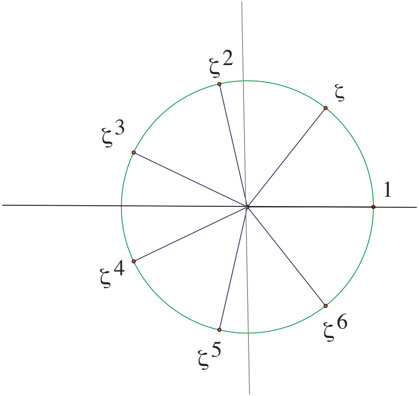

- The six non-real roots come in conjugate pairs.
- So $(\zeta + \zeta^6)$, $(\zeta^2 + \zeta^5)$, and $(\zeta^3 + \zeta^4)$ are real numbers.

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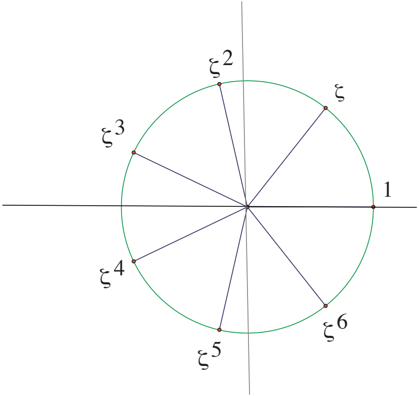

- The six non-real roots come in conjugate pairs.
- So $(\zeta + \zeta^6)$, $(\zeta^2 + \zeta^5)$, and $(\zeta^3 + \zeta^4)$ are real numbers.
- What cubic equation over $\mathbb{R}$ has these three numbers as roots?

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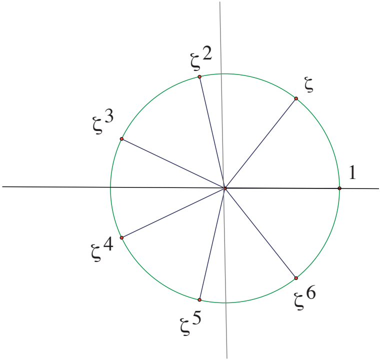
Modeling: Roots of Unity



Let

$$\alpha = \zeta + \zeta^6$$
$$\beta = \zeta^2 + \zeta^5$$
$$\gamma = \zeta^3 + \zeta^4$$

To find an equation satisfied by α, β, and γ, we need to find

- $\alpha + \beta + \gamma$
- $\alpha\beta + \alpha\gamma + \beta\gamma$
- $\alpha\beta\gamma$

One at a time. . .

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Modeling: Roots of Unity

The Sum:

Since $\alpha = \zeta + \zeta^6$, $\beta = \zeta^2 + \zeta^5$, and $\gamma = \zeta^3 + \zeta^4$, we have

$$\alpha + \beta + \gamma = \zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta$$

But

$$x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$$

So,

$$\zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta = -1$$

Modeling: Roots of Unity

The Product:

$$\alpha\beta\gamma = \left(\zeta + \zeta^6\right)\left(\zeta^2 + \zeta^5\right)\left(\zeta^3 + \zeta^4\right)$$

We can get the form of the expansion by expanding

$$\left(z + z^6\right)\left(z^2 + z^5\right)\left(z^3 + z^4\right)$$

Time for a CAS

Modeling: Roots of Unity

So,

$$\left(z+z^6\right)\left(z^2+z^5\right)\left(z^3+z^4\right)=$$

$$z^{15}+z^{14}+z^{12}+z^{11}+z^{10}+z^9+z^7+z^6$$

But if we replace z by ζ, we can replace z^7 by 1...

Modeling: Roots of Unity

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But if we replace z by ζ, we can replace z^7 by 1...
So, if the above expression is written as

$$(z^7-1)q(z)+r(z)$$

then replacing z by ζ will produce $r(\zeta)$

Time for a CAS

CME PROJECT

Modeling: Roots of Unity

So,

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Time for a CAS

So $\alpha\beta\gamma = 1$

Modeling: Roots of Unity

What about the "beast"? Well, $\alpha\beta + \alpha\gamma + \beta\gamma =$

$$\left(\zeta + \zeta^6\right)\left(\zeta^2 + \zeta^5\right) +$$
$$\left(\zeta + \zeta^6\right)\left(\zeta^3 + \zeta^4\right) +$$
$$\left(\zeta^2 + \zeta^5\right)\left(\zeta^3 + \zeta^4\right)$$

Time for a CAS

CME PROJECT

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Modeling: Roots of Unity

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$$\left(\zeta + \zeta^6\right)\left(\zeta^2 + \zeta^5\right) + \left(\zeta + \zeta^6\right)\left(\zeta^3 + \zeta^4\right) + \left(\zeta^2 + \zeta^5\right)\left(\zeta^3 + \zeta^4\right)$$

Time for a CAS

So $\alpha\beta + \alpha\gamma + \beta\gamma = -2$ and our cubic is

$$x^3 + x^2 - 2x - 1 = 0$$

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Modeling: Roots of Unity

- In this informal way, students preview the idea that one can model $\mathbb{Q}(\zeta)$ by "remainder arithmetic" in $\mathbb{Q}(z)$, using $z^7 - 1$ as a divisor.

Modeling: Roots of Unity

- In this informal way, students preview the idea that one can model $\mathbb{Q}(\zeta)$ by "remainder arithmetic" in $\mathbb{Q}(z)$, using $z^7 - 1$ as a divisor.
- In fact, one can use any polynomial that has ζ as a zero—the smallest degree one is

$$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$$

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Modeling: Roots of Unity

- In this informal way, students preview the idea that one can model $\mathbb{Q}(\zeta)$ by “remainder arithmetic” in $\mathbb{Q}(z)$, using $z^7 - 1$ as a divisor.
- In fact, one can use any polynomial that has ζ as a zero—the smallest degree one is

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- This previews Kronecker’s construction of splitting fields for algebraic equations.

Modeling: Roots of Unity

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$$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$$

- This previews Kronecker's construction of splitting fields for algebraic equations.
- And it gives some nice trig identities:

$$8\cos^3\frac{2\pi}{7} + 4\cos^2\frac{2\pi}{7} - 4\cos\frac{2\pi}{7} = 1$$

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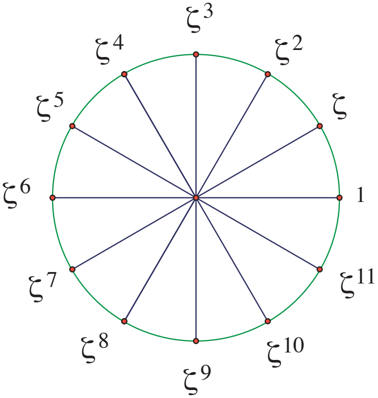
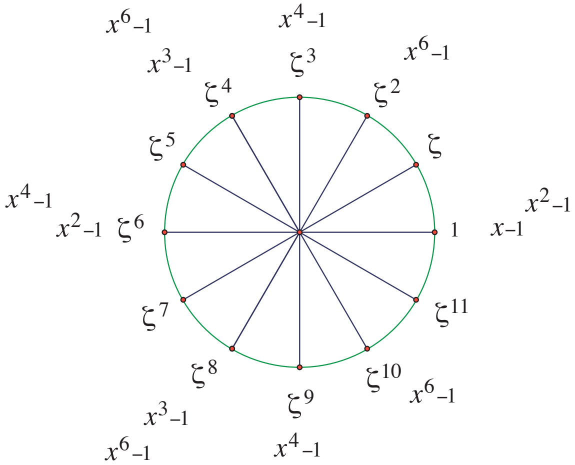
Further Applications: Cyclotomic polynomials

Here are the 12th roots of unity:



Further Applications: Cyclotomic polynomials

Some are roots of $x^n - 1 = 0$ for $n < 12$:



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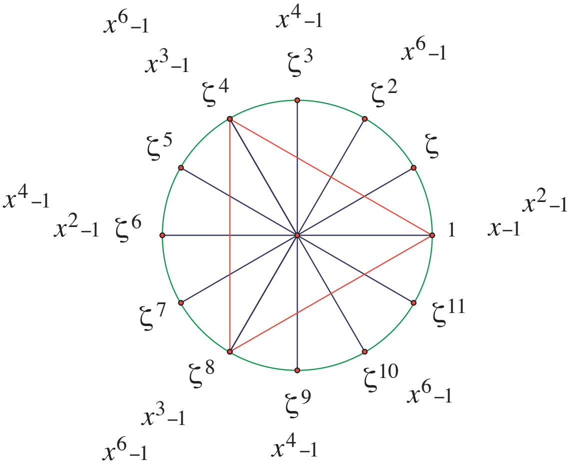
Further Applications: Cyclotomic polynomials

Some are cube roots of unity:



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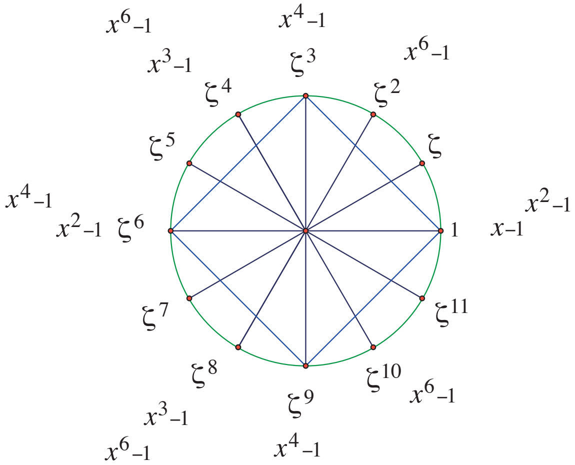
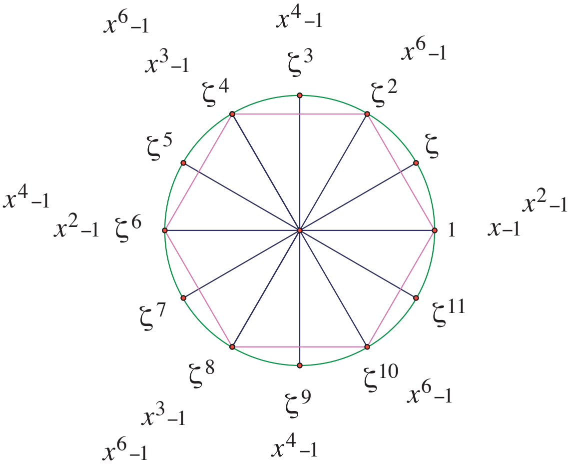
FURTHER APPLICATIONS: CYCLOTOMIC POLYNOMIALS

Some are fourth roots of unity:



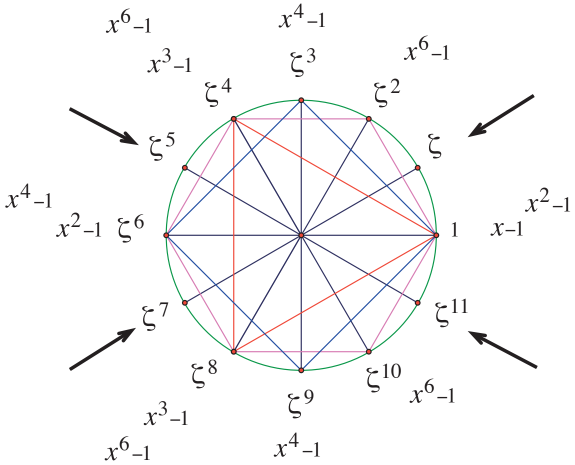
FURTHER APPLICATIONS: CYCLOTOMIC POLYNOMIALS

And some are sixth roots of unity:



Further Applications: Cyclotomic polynomials

Left are the *primitive* 12th roots of unity: ζ, ζ^5, ζ^7, and ζ^{11}.



Scratchpad

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Further Applications: Cyclotomic polynomials

So, the roots of $x^{12} - 1 = 0$ break up like this

$$(x-1) \quad (x+1) \quad (x^2+x+1) \quad (x^2+1) \quad (x^2-x+1) \quad (x^4-x^2+1)$$

Further Applications: Cyclotomic polynomials

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$(x-1)$	$(x+1)$	(x^2+x+1)	(x^2+1)	(x^2-x+1)	(x^4-x^2+1)
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
1	-1	$-\frac{1}{2} \pm \frac{\sqrt{3}}{2}$	$\pm i$	$\frac{1}{2} \pm \frac{\sqrt{3}}{2}$	$\pm\frac{\sqrt{3}}{2} \pm \frac{1}{2}$

CME PROJECT

Further Applications: Cyclotomic polynomials

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\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\zeta^0 & \zeta^6 & \{\zeta^4, \zeta^8\} & \{\zeta^3, \zeta^9\} & \{\zeta^2, \zeta^{10}\} & \{\zeta, \zeta^5, \zeta^7, \zeta^{11}\}
\end{array}$$

Further Applications: Cyclotomic polynomials

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\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\text{first} & \text{second} & \text{third} & \text{fourth} & \text{sixth} & \text{twelvth}
\end{array}
$$

Further Applications: Cyclotomic polynomials

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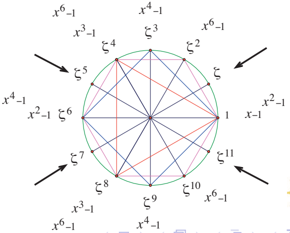
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
first	second	third	fourth	sixth	twelvth

- There is one factor for every divisor d of 12.
- The zeros of the factor for d are the primitive dth roots of 1.
- The primitive 12th roots are ζ^k where $\gcd(k, 12) = 1$



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Further Applications: Cyclotomic polynomials

In general

- If $\zeta_m = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}$, the primitive mth roots of unity are

$$\left\{\zeta_m^k \mid \gcd(k, m) = 1\right\}$$

Further Applications: Cyclotomic polynomials

In general

- If $\zeta_m = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}$, the primitive *m*th roots of unity are

$$\left\{\zeta_m^k \mid \gcd(k,m) = 1\right\}$$

- If

$$\psi_m(x) = \prod_{\substack{1 \le k \le m \\ \gcd(k,m)=1}} \left(x - \zeta_m^k\right)$$

then

- $\psi_m(x)$ has integer coefficients and is irreducible over $\mathbb{Z}$,
- the degree of $\psi_m(x)$ is $\phi(m)$ (Euler's ϕ-function), and...

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Further Applications: Cyclotomic polynomials

The CAS factorization of $x^n - 1$ is precisely

$$x^n - 1 = \prod_{d \mid n} \psi_d(x)$$

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Further Applications: Cyclotomic polynomials

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So, for example, $x^{12} - 1 =$

$$\underset{\psi_1(x)}{\underset{\uparrow}{(x-1)}} \quad \underset{\psi_2(x)}{\underset{\uparrow}{(x+1)}} \quad \underset{\psi_3(x)}{\underset{\uparrow}{(x^2+x+1)}} \quad \underset{\psi_4(x)}{\underset{\uparrow}{(x^2+1)}} \quad \underset{\psi_6(x)}{\underset{\uparrow}{(x^2-x+1)}} \quad \underset{\psi_{12}(x)}{\underset{\uparrow}{(x^4-x^2+1)}}$$

Further Applications: Cyclotomic polynomials

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\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\psi_1(x) & \psi_2(x) & \psi_3(x) & \psi_4(x) & \psi_6(x) & \psi_{12}(x) \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\phi(1)=1 & \phi(2)=1 & \phi(3)=2 & \phi(4)=2 & \phi(6)=2 & \phi(12)=4
\end{array}$$

FURTHER APPLICATIONS: CYCLOTOMIC POLYNOMIALS

Consequences:

- The partition of n given by the factors is ▸ Scratchpad

$$\{\phi(d) \mid d \text{ is a factor of } n\}$$

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Further Applications: Cyclotomic polynomials

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$$\sum_{d|n} \phi(d) = n$$

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Further Applications: Cyclotomic polynomials

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$$\{\phi(d) \mid d \text{ is a factor of } n\}$$

- As a bonus, we get the famous

$$\sum_{d|n} \phi(d) = n$$

- We can compute the irreducible factors recursively from $x^n - 1 = \prod_{d \mid n} \psi_d(x)$: Scratchpad

$$\psi_n(x) = \frac{x^n - 1}{\prod_{\substack{d \mid n \\ d<n}} \psi_d(x)}$$

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Further Applications: Cyclotomic polynomials

A wonderful theorem:

If k is odd, and if $p_1 < p_2 < \cdots < p_k$ is a "front-loaded" sequence of primes, so that the sum of the first two in the sequence is greater than the last, and if n is the product of all the primes in the sequence, then $\psi_n(x)$ has $-k+1$ and $-k+2$ as coefficients.

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Further Applications: Cyclotomic polynomials

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Example: since $105 = 3 \cdot 5 \cdot 7$, and $\{3, 5, 7\}$ is a front-loaded sequence of length 3, $\psi_{105}(x)$ has a coefficient of -2. In fact, it's the coefficient of x^7. Scratchpad

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Further Applications: Cyclotomic polynomials

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More details in "On Coefficients of Cyclotomic Polynomials." Jiro Suzuki: Proc. Japan Acad., Ser. A,1987.

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Further Applications: Cyclotomic polynomials

To get a coefficient of -3, the theorem demands that we look at ψ_n where $n = 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 = 1062347$.

FURTHER APPLICATIONS: CYCLOTOMIC POLYNOMIALS

To get a coefficient of -3, the theorem demands that we look at ψ_n where $n = 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 = 1062347$.

This is where the trouble started.

Scratchpad

Further Applications: Cyclotomic polynomials

Hi Cleve,

I'm giving a talk later this week at the USACAS conference, and I want to talk about the fact that coefficients of cyclotomic polynomials can be made as large as you like. One way is to take the *n*th polynomial where n is a product of k distinct primes (k odd) so that the sum of the first two is larger than the last. The first of these is $k = 105 = 3 \cdot 5 \cdot 7$. The first length 5 sequence is $n = 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$, but neither the nSpire nor Mathematica will generate the polynomial (or its coefficient list).

Can MATLAB do it?

Al

Further Applications: Cyclotomic polynomials

By itself, MATLAB cannot handle integers larger than 2^{52}. With the Symbolic Toolbox, MATLAB can handle arbitrarily large integers. Without Mathematica's function, how would you generate $\psi_n(x)$?

—Cleve

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FURTHER APPLICATIONS: CYCLOTOMIC POLYNOMIALS

Well, Mathematica has a built in “Cyclotomic” command, so I typed

```
Cyclotomic[11*13*17*19*23, x]
```

And it just chugs. On the nSpire, I used the fact that if $\psi_n(x)$ is the nth cyclotomic poly, then

$$\psi_n(x) = \frac{x^n - 1}{\prod_{\substack{d \mid n \\ d<n}} \psi_d(x)}$$

There’s also a recurrence that uses the Möebius function. Or,

$$\prod (x - e^{\frac{2k\pi i}{n}})$$

Where k ranges over the integers relatively prime to n

—Al

Further Applications: Cyclotomic polynomials

Attached are two programs. I think “cyclo” is what we want, but it never finishes. “cyclo2” is a simplified version. It doesn’t finish either.

I’m actually testing the next version of the Symbolic Toolbox. We haven’t released it yet. I’m checking with the guys working on this next version, and I’m going to let “cyclo” run for awhile—maybe overnight.

I have another idea to try—more later...

—Cleve

Further Applications: Cyclotomic polynomials

The next day

⋮

Further Applications: Cyclotomic polynomials

Al—

For $n = 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 = 1062347$, the degree of $\psi_n(x)$ is 760320. The coefficients range from -1749 to $+1694$. There are 11804 zero coefficients. The average coefficient magnitude is 409.9.

My MATLAB program is attached. It uses the Symbolic Toolbox for the number theoretic functions moebius(n) and divisors(n), but *not for any polynomial manipulations*. Polynomials are represented as MATLAB vectors with integer elements. Polynomial multiplication and division is done by MATLAB vector convolution and deconvolution.

⋮

Further Applications: Cyclotomic polynomials

$\psi_n(x)$ is computed from the ratio of two polynomials, a numerator of degree 1105920 and a denominator of degree 345600. It takes about 6 minutes on my laptop to compute the numerator and denominator and then about $2\frac{1}{2}$ hours to compute their ratio using only the deconvolution.

I've saved the results. Anything else you'd like to know?

—Cleve

$$\psi_n(x) = \prod_{d|n} \left(1 - x^{\frac{n}{d}}\right)^{\mu(d)}$$

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Further Applications: Cyclotomic polynomials

What was it the theorem said? That there would be a coefficient of -4??

Looks like the actual data slaughtered -4. I am curious whether that specific coefficient AI mentioned is -4.

It reminds me of the early 20th century theorem that any odd number could be written as the sum of no more than 20,000 primes. Later that requirement got reduced to 3!

—Bowen

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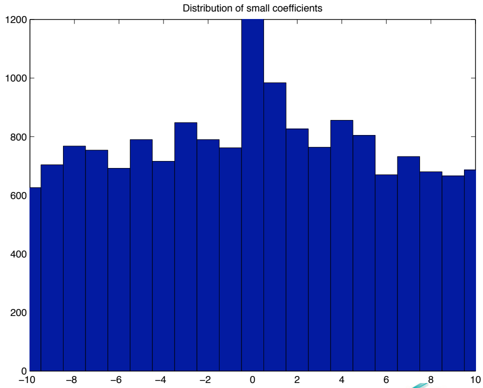
Further Applications: Cyclotomic polynomials

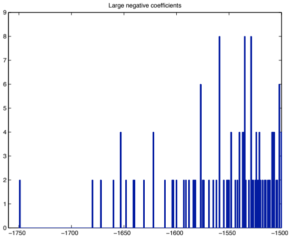
```
length(find(p==4))
ans = 856
length(find(p==-4))
ans = 716
```

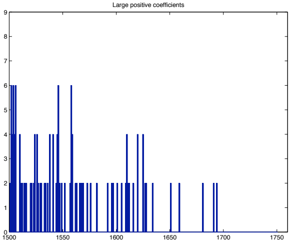
You might find the attached histograms interesting.

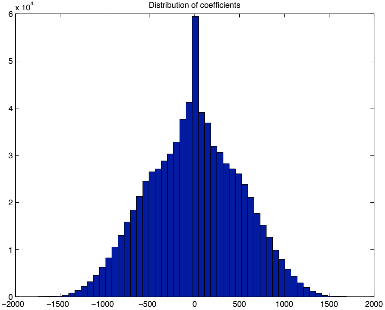
— Cleve



Further Applications: Cyclotomic polynomials







From a computational perspective

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From a computational perspective

- . . . talking to a live mathematician is very different from talking to a CAS.
- It's important to point out that I did the heavy lifting *without* using the CAS available in the Symbolic Toolbox.
- By the way, the convolutions and, especially, the deconvolution can be done in a few minutes instead of a few hours using FFTs, but roundoff errors get in the way. For $n = 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$ the computed results are contaminated enough that rounding them to integers produces some coefficients that are off by ± 1.

—Cleve

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From a mathematical perspective

- Mathematical objects are *real*.

From a mathematical perspective

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  - Mathematical phenomena exhibit all of the intricate and textured features present in physical phenomena.

FROM A MATHEMATICAL PERSPECTIVE

- Mathematical objects are *real*.
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- Mathematical thinking is a wonderful thing.

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From a mathematical perspective

- Mathematical objects are *real*.
  - Mathematical phenomena exhibit all of the intricate and textured features present in physical phenomena.
- Mathematical thinking is a wonderful thing.
  - The human mind can establish facts about mathematical objects even when the objects are difficult (or impossible) to write down explicitly.

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What I had planned to talk about

The theory of equations

Given a polynomial equation like

$$x^7 + 12x^6 + 15x^5 - 114x^4 - 37x^3 + 144x^2 - 51x + 270 = 0 \quad (*)$$

Find—without solving $(*)$—an equation whose roots are

- three times the roots of $(*)$.
- three more than the roots of $(*)$.
- the negatives of the roots of $(*)$.
- the squares of the roots of $(*)$.
- the *n*th powers of the roots of $(*)$.

. . . maybe next time.

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USING A CAS IN HIGH SCHOOL: OTHER EXAMPLES

The CME Project uses a CAS to

1. Experiment with algebra: Chebyshev polynomials

Using a CAS in high school: other examples

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2. Reduce computational overhead: Lagrange interpolation and Newton's Difference Formula

Using a CAS in high school: other examples

The CME Project uses a CAS to

1. Experiment with algebra: Chebyshev polynomials
2. Reduce computational overhead: Lagrange interpolation and Newton's Difference Formula
3. Use polynomials as modeling tools: Generating functions

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For More Information

- For the program:
  - `www.pearsonschool.org/cme`
  - Al Cuoco (acuoco@edc.org)
- For summer workshops
  - `www.edc.org/cmeproject`
  - Melody Hachey (mhachey@edc.org)
  - Sarah Sword (ssword@edc.org)

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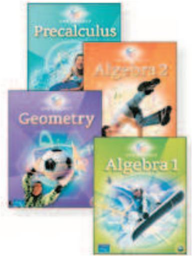
AVAILABILITY



- Algebra 1, Geometry, and Algebra 2
  - **Now**
- Precalculus
  - **This Summer**

Workshops this summer: August 4–8.
Contact Melody Hachey (`mhachey@edc.org`)

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