DYNAMIC GEOMETRY AS A TOOL TO DEVELOP ANALYTIC THINKING

PROMOTING MATHEMATICAL HABITS OF MIND IN HIGH SCHOOL

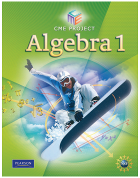
Al Cuoco
Center for Mathematics Education (CME), EDC

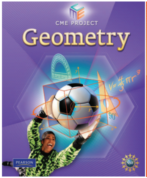
Sapienza, Università di Roma, February 12, 2009

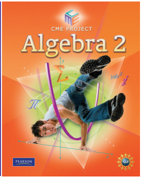
Slides available at
`www.edc.org/cmeproject`

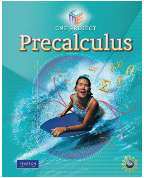
Ideas taken from

- The *CME Project*, a four-year high school curriculum

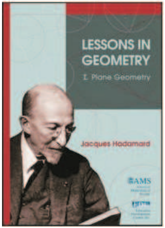








- Hadamard's *Lessons in Geometry*



Our use of technology in CME programs

We use

- A “function modeling language” so that students can learn to
  - build computational models of mathematical functions
  - encapsulate processes and reify functions
  - develop “procedural abstraction” (Abelson-Sussman)
- A spreadsheet so that students can learn to
  - tabulate functions
  - construct difference tables
  - develop “data abstraction” (Abelson-Sussman)

Our use of technology in CME programs

- A computer algebra system so that students can learn to
  - reason about formal calculations
  - experiment with polynomials and other algebraic expressions
  - build computational models of algebraic structures
  - develop algebraic habits of mind
- A dynamic geometry environment so that students can learn to
  - experiment with geometric phenomena
  - build computational models of continuously changing systems
  - define functions on geometric constructions
  - develop analytic habits of mind

Goal for Today

Use classical geometry problems to show how dynamic geometry environments can be used to . . .

- approach the problems,
- extend the problems,
- build some mathematical habits that underlie analysis

Outline

6 RESOURCES

- The CME Project
- Hadamard's "Lessons in Geometry"

7 CONTACTS

- For More Information

The Habits of Mind Approach

Mathematics constitutes one of the most ancient and noble intellectual traditions of humanity. It is an enabling discipline for all of science and technology, providing powerful tools for analytical thought as well as the concepts and language for precise quantitative description of the world around us. ***It affords knowledge and reasoning of extraordinary subtlety and beauty, even at the most elementary levels.***

— Hyman Bass, Mathematics Study Panel, 2002

The Habits of Mind Approach

What mathematicians most wanted and needed from me was ***to learn my ways of thinking****, and not in fact to learn my proof of the geometrization conjecture for Haken manifolds.*

— William Thurston

On Proof and Progress in Mathematics

Our Fundamental Organizing Principle

The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more importantly, from developing the ways of thinking—the ***habits of mind****—used to create the results.*

—*The CME Project* Teacher Guide, 2008

General Mathematical Habits

- Performing thought experiments
- Finding and explaining patterns
- Creating and using representations
- Generalizing from examples
- Expecting mathematics to make sense

Algebraic Habits of Mind

- Seeking regularity in repeated calculations
- “Delayed evaluation”—seeking form in calculations
- “Chunking”—changing variables in order to hide complexity
- Reasoning about and picturing calculations and operations
- Extending operations to preserve rules for calculating
- Purposefully transforming and interpreting expressions
- Seeking and specifying structural similarities

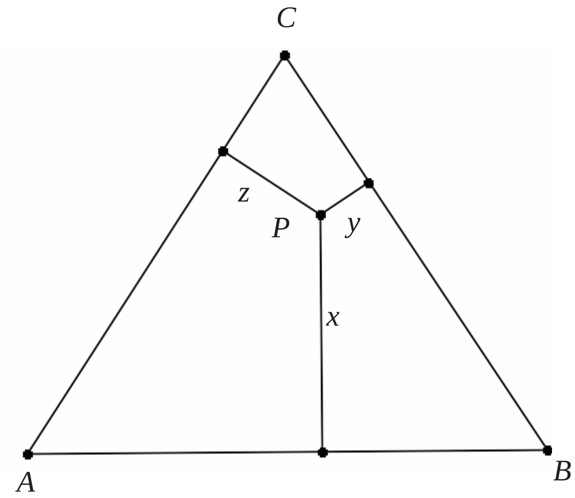
Analytic/Geometric Habits of Mind

- Reasoning by continuity
- Seeking geometric invariants
- Looking at extreme cases
- Passing to the limit
- Modeling geometric phenomena with continuous functions

Today's talk concerns these analytic habits

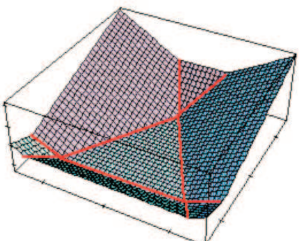
Example: Invariants in Triangles

Rich's function



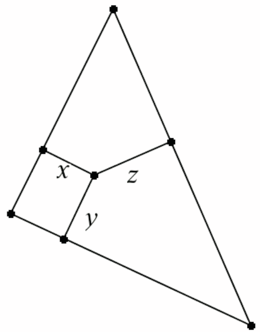
EXTENSION

- **What if the point is outside the triangle?**
  - This is related to **Hadamard, Problem 42**: In an isosceles triangle, show that the sum of the distances from any point on the base to the other two sides is constant.
- **What is the surface plot of Rich's function?**



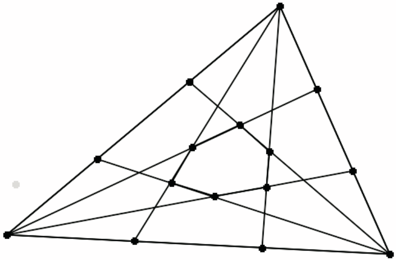
Extension

What if the triangle is not equilateral?



EXAMPLE: INVARIANTS IN TRIANGLES

MARION WALTER'S DISCOVERY



A 9th grade student in Maryland (near Washington DC) got interested in the problem. . .

Ryan's Journal

"One day, my geometry teacher took our class across the hall to our school's computer lab. He wanted us to get familiar with the use of the computers. He did this by having us 'discover' Marion Walter's Theorem on our own. He told us how to draw the triangle, trisect the sides, and draw the hexagon in the middle. It was our job to find something 'neat' about the measurements of the shapes. When I found Marion Walter's Theorem, I got curious, and wanted to see if the same thing worked with any shapes other than triangles."

Ryan's Journal

"This is where all the hard work began. Not having a computer of my own, I was forced to use my school's computers after school. I attempted to trisect the sides of the square, and see if there was any special relationship there. At first I thought there was. I can't remember exactly what the value was, but it did seem that there was a constant ratio between the square and the octagon that was in the middle of the square. At that point, I really thought I was on to something."

Ryan's Journal

"So, the next thing I did was try the same thing with a pentagon. I trisected the sides, compared areas between the pentagon and the now 10-sided figure inside. . . . There was no constant ratio. I was upset, because at this point I had spent maybe a week or two, every day after school, working on this thing, and now I had hit a dead end. But I didn't give up."

Ryan's Journal

"This time I concentrated only on triangles, and no other shape. Trisecting it was the whole basis behind Marion Walter's Theorem. . . . I started 5-secting, 7-secting . . . the triangle. When I compared the area of the triangle and area of the internal hexagon, I noticed a constant ratio between the two. And for every example I tried (odd number 'secting' per side only), I was able to find a ratio that was constant no matter how the triangle was altered in size."

Ryan's Journal

"The next step was to find a relationship between the number of sections per side and the ratio between the areas of the triangle and hexagon. Using the regression functions on a regular scientific calculator, I was able to do just that, and came up with the formula

$$y = \frac{9n^2 - 1}{8}$$

where n is the number of sections per side, and y is how many times bigger the area of the triangle is than the area of the hexagon. It took a lot of time and effort, but it was worth it. And before I can take credit for my theorem, I must thank Marion for creating her theorem to begin with; without hers, mine never would have come around."

Eventually, Ryan and his teachers found a proof for his formula.

WHAT KIDS CAN DO. . .

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SOMETHING NEW IN A TRIANGLE

Patapsco High student's hunch points to theorem

By Mary Maushard
Sun Staff Writer

Ryan Morgan would have gotten an "A" in geometry even if he hadn't unearthed a mathematical treasure.

But the persistent Patapsco High School sophomore pushed a hunch into a theory. He calls it Morgan's Conjecture, and is hoping it will soon be Morgan's Theorem.

In geometric circles, developing a theorem is a big deal — especially if you're only 15.

Ryan's teacher at Patapsco High, Frank Nowosielski, has been teaching 20 years and has never had a student discover a theorem — a mathematical statement that can be proved universally true.

Towson State University math professor Robert B. Hanson never had a high school student present a possible theorem to his faculty seminar — until Ryan did it last spring.

"Ryan's really done something pretty fantastic," said Mr. Nowosielski, who taught Ryan's ninth-grade geometry class for gifted and talented students las' year and now teaches at the Carver Center for Arts and Technology in Towson.

"How many kids in the world have done this? He saw something and he didn't quit. He's a special kid," Mr. Nowosielski said.

What did Ryan see?

Initially, he sa v a triangle, each side divided into thirds. Lines drawn from those segments to the vertices (the corners) formed a hexagon inside the triangle. The area of the hexagon is one-tenth the area of the triangle. This is known as Marion's

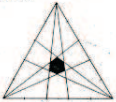
See **THEOREM**, 1&A

JED KIRSCHBAUM/SUN STAFF PHOTO

Ryan Morgan worked many days after school in the computer lab to develop his conjecture, which is displayed on the screen.

MORGAN'S CONJECTURE

Developed by Ryan Morgan, 15, of Patapsco High School:

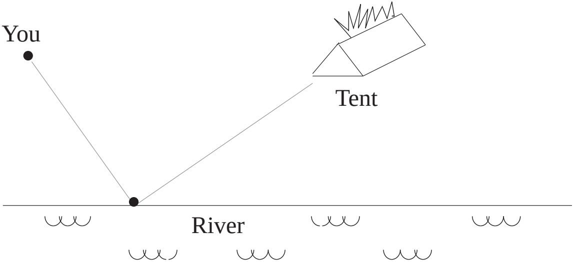


When the sides of a triangle are n-sected, and n represents any odd integer greater than 1, and segments are drawn from the vertices to these new points, there will be a hexagon present in the interior of the triangle (shaded area). There will always be a constant ratio between the area of the hexagon to the area of the original triangle.

SUN STAFF GRAPHIC

THE "BURNING TENT" PROBLEM

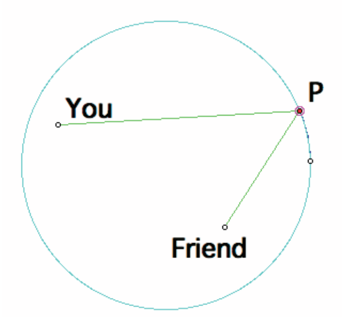
Hadamard: PROBLEMS 13, 14, 15



Three ways to think about it

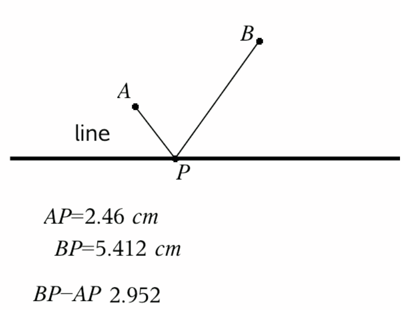
Extension

You and a friend are in a circular swimming pool. You want to swim to the edge, drop off your empty glass, and swim back to your friend. What point on the edge of the pool minimizes the amount of swimming?



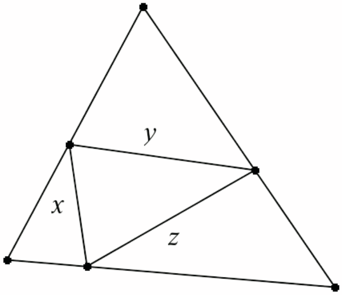
EXTENSION

Hadamard, Problem 15: On a given line, find a point with the property that the *difference* of its distances to two given points is as *large* as possible.



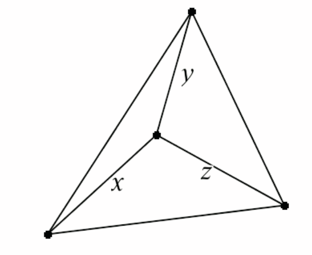
Minimum Paths

A Problem of Fagnano: Of all triangles inscribed in a given triangle, which one has the smallest perimeter?



Minimum Paths

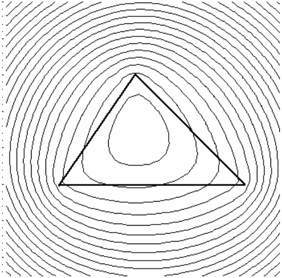
The Airport Problem: Three cities get together to build an airport.



Where should they put it to minimize the cost of new roads?

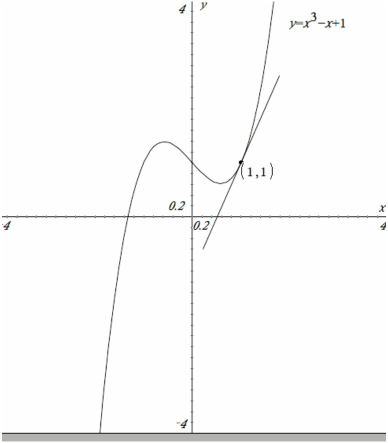
EXTENSION

What are the contour lines for the “airport function”?



Tangents to Graphs

- What does it mean to say that a line is tangent to the graph of the equation $y = x^3 - x + 1$ (or any other equation)?



Tangents to Graphs

- *The classic thought experiment*:
  - a line tangent to the graph of $y = f(x) = x^3 - x + 1$ at $(1, 1)$ is the limit of secant lines that connect $(1, 1)$ to nearby points on the graph.
- What does the algebra say?

Tangents to Graphs

- $(x-1)(x-2) = x^2 - 3x + 2$, a quadratic, so
- the remainder when $f(x)$ is divided by $(x-1)(x-2)$ is a *linear* polynomial, $r(x)$

$$\begin{array}{ccccccccc} x^3 - x + 1 & = & x^2 - 3x + 2 & \cdot & \text{something} & + & (ax+b) \\ \uparrow & & \uparrow & \uparrow & \uparrow & & \uparrow \\ f(x) & = & (x-1)(x-2) & \cdot & q(x) & + & r(x) \end{array}$$

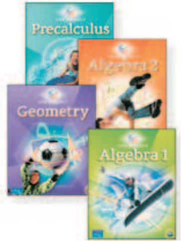
- Also, $r(1) = f(1)$ and $r(2) = f(2)$,
- so the equation of the secant if the graph of $y = f(x)$ between $x = 1$ and $x = 2$ is $y = r(x)$.
- We can find $r(x)$ by hand…or with a little help from a friend.

Conclusion

- There are ways of thinking indigenous to analysis.
- These mathematical habits can be developed in classical geometric contexts.
- Dynamic geometry can help students develop these habits.

The CME Project: Brief Overview

- Funded by the National Science Foundation
- Follows the traditional American course structure
- Uses Texas Instruments TI-Nspire handheld technology
  - “Function modeling language”
  - Spreadsheet
  - Dynamic geometry and graphs
  - Computer algebra system
- **Core organizing principle**: mathematical habits of mind



EDC

Additional Core Principles

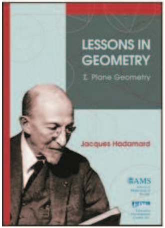
- **Textured emphasis.** We focus on matters of mathematical substance, being careful to separate them from convention and vocabulary. Even our practice problems are designed so that they have a larger mathematical point.
- **General purpose tools.** The methods and habits that students develop in high school should serve them well in their later work in mathematics and in their post-secondary endeavors.
- **Experience before formality.** Worked-out examples and careful definitions are important, but students need to grapple with ideas and problems *before* they are brought to closure.

Additional Core Principles, continued

- **The role of applications.** What matters is *how* mathematics is applied, not *where* it is applied.
- **A mathematical community.** Our writers, field testers, reviewers, and advisors come from all parts of the mathematics community: teachers, mathematicians, education researchers, technology developers. . . .
- **Connect school mathematics to the discipline.** Every chapter, lesson, problem, and example is written with an eye towards how it fits into the landscape of mathematics as a scientific discipline.

Hadamard's "Lessons in Geometry"

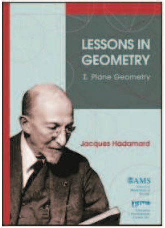
From the Preface



This is a book in the tradition of Euclidean synthetic geometry written by one of the twentieth century's great mathematicians. The original audience was pre-college teachers, but it is useful as well to gifted high school students and college students, and, in particular, to mathematics majors interested in geometry from a more advanced standpoint.

HADAMARD'S "LESSONS IN GEOMETRY"

FROM THE PREFACE

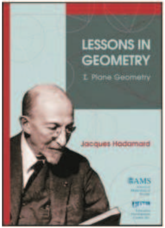


The text starts where Euclid starts, and covers all the basics of plane Euclidean geometry. But this text does much more. It is at once pleasingly classic and surprisingly modern. The problems (more than 450 of them) are well-suited to exploration using the modern tools of dynamic geometry software. For this reason, the present edition includes a CD of dynamic solutions to select problems, created using Texas Instruments' TI-Nspire™ Learning Software.

HADAMARD'S "LESSONS IN GEOMETRY"

FROM THE PREFACE

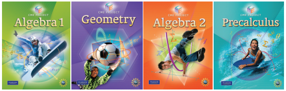
The TI-Nspire™ documents demonstrate connections among problems and. . . will allow the reader to explore and interact with Hadamard's Geometry in new ways. The material also includes introductions to several advanced topics. The exposition is spare, giving only the minimal background needed for a student to explore these topics. Much of the value of the book lies in the problems, whose solutions open worlds to the engaged reader.



For More Information

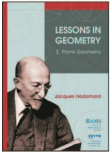
- For more information about the *CME Project*, including this presentation, see

www.edc.org/cmeproject



- For more information about *Lessons in Geometry*, see

http://www.ams.org/bookstore-getitem/item=mbk-57



Grazie molto

Al Cuoco

Center for Mathematics Education

Education Development Center

acuoco@edc.org