Reasoning and Sense Making with Technology

Some Examples from Algebra and Functions

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Outline

THE MAIN POINT

There are three uses of “this kind” of technology that can help students build ideas:

1. Reduce computational overhead
2. Construct and perform experiments
3. Build computational models of mathematical objects

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A STANDARD PROBLEM

Find a function that agrees with this table.

INPUT	OUTPUT
0	1
1	3
2	5
3	7
4	9

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What would you do if . . .

INPUT	OUTPUT
0	1
1	3
2	5
3	7
4	9

Sasha says, "I know, I know, it's

$$f(n) = n^5 - 10n^4 + 35n^3 - 50n^2 + 26n + 1,"$$

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Variation 1

Find some polynomial functions that agree with this table.

INPUT	OUTPUT
0	1
1	3
2	5
3	7
4	9

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Variation 1

- Suppose you have two functions, f and g that agree on
$$\{0, 1, 2, 3, 4\}$$
- If f and g are polynomial functions, then $g - f$ is a polynomial function with zeros at $\{0, 1, 2, 3, 4\}$.
- By the factor theorem, $g - f$ has as factors
$$x, x - 1, x - 2, x - 3, x - 4$$
- Hence
$(g - f)(x) = \text{something} \cdot x(x - 1)(x - 2)(x - 3)(x - 4)$
- and

$$g(x) = f(x) + k \cdot x(x - 1)(x - 2)(x - 3)(x - 4)$$

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Variation 2

Find a function that agrees with this table.

INPUT	OUTPUT
0	1
1	3
2	5
3	7
4	9

Two models

Now we have two models :

$$f(n) = 2n + 1 \qquad g(n) = \begin{cases} 1 & n = 0 \\ g(n-1) + 2 & n > 0 \end{cases}$$

$$f \stackrel{?}{=} g$$

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Two models

$$f(n) = 2n + 1 \qquad g(n) = \begin{cases} 1 & n = 0 \\ g(n-1) + 2 & n > 0 \end{cases}$$

Suppose on your handheld, $f(n) = g(n)$ for $0 \le n \le 64$, but $f(65)$ reports 131 and $g(65)$ reports an error.

$$\begin{aligned} g(65) &= g(64) + 2 && \text{(this is how } g \text{ is defined)} \\ &= f(64) + 2 && \text{(CSS)} \\ &= (2 \cdot 64 + 1) + 2 && \text{(this is how } f \text{ is defined)} \\ &= (2 \cdot 64 + 2) + 1 && \text{(arithmetic)} \\ &= (2 \cdot 65) + 1 && \text{(more arithmetic)} \\ &= f(65) && \text{(this is how } f \text{ is defined)} \end{aligned}$$

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TWO MODELS

$$f(n) = 2n + 1 \qquad g(n) = \begin{cases} 1 & n = 0 \\ g(n-1) + 2 & n > 0 \end{cases}$$

Suppose on your handheld, $f(n) = g(n)$ for $0 \le n \le 254$, but $f(255)$ reports 510 and $g(255)$ reports an error.

$$\begin{aligned} g(255) &= g(254) + 2 && \text{(this is how } g \text{ is defined)} \\ &= f(254) + 2 && \text{(CSS)} \\ &= (2 \cdot 254 + 1) + 2 && \text{(this is how } f \text{ is defined)} \\ &= (2 \cdot 254 + 2) + 1 && \text{(arithmetic)} \\ &= (2 \cdot 255) + 1 && \text{(more arithmetic)} \\ &= f(255) && \text{(this is how } f \text{ is defined)} \end{aligned}$$

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Two models

$$f(n) = 2n + 1 \qquad g(n) = \begin{cases} 1 & n = 0 \\ g(n-1) + 2 & n > 0 \end{cases}$$

Suppose on your (virtual) handheld, $f(n) = g(n)$ for $0 \leq n \leq k - 1$, but $f(k)$ reports $2k + 1$ and $g(k)$ reports an error.

$$\begin{aligned} g(k) &= g(k-1) + 2 \quad \text{(this is how } g \text{ is defined)} \\ &= f(k-1) + 2 \quad \text{(VCSS)} \\ &= (2 \cdot (k-1) + 1) + 2 \quad \text{(this is how } f \text{ is defined)} \\ &= (2 \cdot (k-1) + 2) + 1 \quad \text{(arithmetic)} \\ &= (2 \cdot k) + 1 \quad \text{(algebra)} \\ &= f(k) \quad \text{(this is how } f \text{ is defined)} \end{aligned}$$

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Try Some

- Pick your favorite table from the first page of the handout.
  - *Don't pick 1a.*
- Find a closed form and a recursive model that agrees with your table.
- Are your two models equal on $\mathbb{Z}^{\geq 0}$?
  - If not, find a place where they disagree.
  - If so, prove it.

A 2-TERM RECURRENCE

Experiment with this puppy:

$$g(n) = \begin{cases} 2 & n = 0 \\ 2 & n = 1 \\ 2g(n-1) + 3g(n-2) & n > 1 \end{cases}$$

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A MIXED METHODS

How about this one?

$$h(n) = \begin{cases} 2 & n = 0 \\ 3h(n-1) - 2 & n > 1 \end{cases}$$

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MOVING ON

You want to buy a car that costs \$25000, and you can put \$1000 down. The annual interest rate is 5%. What monthly payment would let you own the car after 48 months?

Hint: What you owe at the end of a month is what you owed at the start of the month, multiplied by $1 + \frac{.05}{12}$, minus the monthly payment.

- Write a function $b(n, m)$ that gives the balance on the loan at the end of n months, with a monthly payment of m.
- Then find the m that makes $b(48, m) = 0$.
- Is there a closed form for b?

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THANKS

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