Questions for the Session

- How should the mathematical practices change the culture of our classes?
- How are they different from past practices?
- And what is the role of technology in making such practices a reality?

Geez. . .

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An Example

Connecting Practice to Content in Algebra

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Outline

Two Problems

(1) I'm thinking of a number. When I divide it by 3, the remainder is 2. When I divide it by 5, the remainder is 3. And when I divide it by 7, the remainder is 1. What's my number?

(2) Find a polynomial of smallest degree that agrees with this table:

Input	Output
3	16
5	42
7	84

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The Notion of Mathematical Practice

What mathematicians most wanted and needed from me was to learn my ways of thinking, and not in fact to learn my proof of the geometrization conjecture for Haken manifolds.

— William Thurston
"On Proof and Progress in Mathematics."
Bulletin of the American Mathematical Society, 1994

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The Notion of Mathematical Practice

The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more importantly, from developing the ways of thinking—the habits of mind—used to create the results.

—Al Cuoco, Paul Goldenberg, & June Mark
"Habits of Mind: An Organizing Principle for High School Curricula." *The Journal of Mathematical Behavior*, 1996.

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The Notion of Mathematical Practice

It will be helpful to name and (at least partially) specify some of the things—practices, dispositions, sensibilities, habits of mind—entailed in doing mathematics. . . . These are things that mathematicians typically do when they do mathematics. At the same time most of these things, suitably interpreted or adapted, could apply usefully to elementary mathematics no less than to research.

—Hyman Bass
"A Vignette of Doing Mathematics." *The Montana Mathematics Enthusiast*, 2011.

The Notion of Mathematical Practice

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise. . . . Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

—CCSS, 2010

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Structural Similarity

- Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. (CCSS, A-APR.1)
- . . . the standards call for attention to the structural similarities between polynomials and integers. The study of these similarities can be deepened in Algebra II: Like integers, polynomials have a division algorithm, and division of polynomials can be used to understand the factor theorem, to transform rational expressions, to help solve equations, and to factor polynomials. (PARCC *Model Content Frameworks*, p57, 2011)

They "Calculate the Same."

$$\begin{array}{r} 10 \\ 6\overline{)\,62} \\ 60 \\ \hline 2 \end{array}$$

$$\begin{array}{r} 3 \\ 2x^2-x-1\overline{)\,6x^2+\ x-1} \\ 6x^2-3x-3 \\ \hline 4x+2 \end{array}$$

Let's Start with Integers

Theorem (Euclid): If b is divided by a and the remainder is r, then

$$\gcd(a, b) = \gcd(r, a)$$

$$\begin{array}{r|l} & 10 \\ \hline 6) & 62 \\ & 60 \\ \hline & 2 \end{array}$$

$$62 = 6 \cdot 10 + 2 \quad \text{and} \quad 62 - 6 \cdot 10 = 2$$

$$\gcd(6, 62) = \gcd(2, 6)$$

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LET'S FIND $\gcd(216, 3162)$

$$
\begin{array}{r}
14 \\
216 \overline{)\ 3162} \\
3024 \\
\hline
138
\end{array}
$$

$$\text{mod}\,(3162, 216) = 138 \quad \text{so...}$$

$$\gcd(216, 3162) = \gcd(138, 216)$$

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LET'S FIND $\gcd(138, 216)$

$$\begin{array}{r} 14 \\ 216\overline{)\,3162} \\ 3024 \\ \hline 138 \end{array}$$

$$\begin{array}{r} 1 \\ 138\overline{)\,216} \\ 138 \\ \hline 78 \end{array}$$

$$\bmod(216, 138) = 78 \quad \text{so}\ldots$$

$$\gcd(138, 216) = \gcd(78, 138)$$

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Keep Up the Good Work: Now gcd(78, 138)

$$\begin{array}{r} 14 \\ 216 \,\overline{)\,3162} \\ 3024 \\ \hline 138 \end{array} \qquad \begin{array}{r} 1 \\ 138 \,\overline{)\,216} \\ 138 \\ \hline 78 \end{array} \qquad \begin{array}{r} 1 \\ 78 \,\overline{)\,138} \\ 78 \\ \hline 60 \end{array}$$

$\text{mod}\,(138, 78) = 60$ so...

$\gcd(78, 138) = \gcd(60, 78)$

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A Rhythm Emerges

```
       14
216) 3162
     3024     1
      138) 216
           138     1
            78) 138
                 78    1
                 60) 78
                     60    3
                     18) 60
                         54    3
                          6) 18
                             18
                              0
```

$$\gcd(216, 3162) = \gcd(138, 216)$$

$$\gcd(138, 216) = \gcd(78, 138)$$

$$\gcd(78, 138) = \gcd(60, 78)$$

$$\gcd(60, 78) = \gcd(18, 60)$$

$$\gcd(18, 60) = \gcd(6, 18)$$

$$\gcd(6, 18) = \gcd(0, 6) = 6$$

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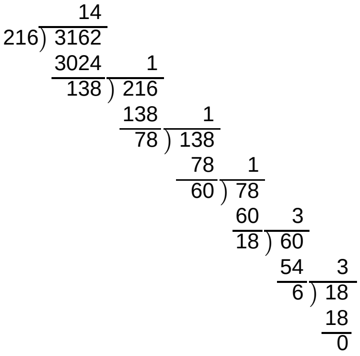
ABSTRACTING REGULARITY



$$\begin{aligned}
\gcd(216, 3162) &= \gcd(138, 216)\\
\gcd(138, 216) &= \gcd(78, 138)\\
\gcd(78, 138) &= \gcd(60, 78)\\
\gcd(60, 78) &= \gcd(18, 60)\\
\gcd(18, 60) &= \gcd(6, 18)\\
\gcd(6, 18) &= \gcd(0, 6) = 6
\end{aligned}$$

Squeezing this into precise language, we have a function:

$$g(a, b) = \begin{cases} b & a = 0 \\ g\left(\text{mod}(b, a), a\right) & \text{otherwise} \end{cases}$$

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Working It Backwards

$$\begin{array}{r} 14 \\ 216\overline{)\,3162} \\ 3024 \\ \hline 138 \end{array}
\quad
\begin{array}{r} 1 \\ 138\overline{)\,216} \\ 138 \\ \hline 78 \end{array}
\quad
\begin{array}{r} 1 \\ 78\overline{)\,138} \\ 78 \\ \hline 60 \end{array}
\quad
\begin{array}{r} 1 \\ 60\overline{)\,78} \\ 60 \\ \hline 18 \end{array}
\quad
\begin{array}{r} 3 \\ 18\overline{)\,60} \\ 54 \\ \hline 6 \end{array}
\quad
\begin{array}{r} 3 \\ 6\overline{)\,18} \\ 18 \\ \hline 0 \end{array}$$

$$138 = 3162 - 14 \cdot 216$$

$$78 = 216 - 1 \cdot 138$$

$$60 = 138 - 1 \cdot 78$$

$$18 = 78 - 1 \cdot 60$$

$$6 = 60 - 3 \cdot 18$$

Abstracting Regularity (Again)

Rewrite $b = aq + r$ as $r = -qa + b$:

$r(b,a)$	$=$	$-q(b,a)\cdot a$	$+$	b
138	$=$	$-14\cdot 216$	$+$	3162
78	$=$	$-1\cdot 138$	$+$	216
60	$=$	$-1\cdot 78$	$+$	138
18	$=$	$-1\cdot 60$	$+$	78
6	$=$	$-3\cdot 18$	$+$	60

$\gcd(a,b)$	$=$	$= sa + tb$
6	$= -3\cdot 18 + 60$	$= -3\cdot 18 + 60$
	$= -3(-1\cdot 60 + 78) + 60$	$= 4\cdot 60 - 3\cdot 78$
	$= 4(-1\cdot 78 + 138) - 3\cdot 78$	$= -7\cdot 78 + 4\cdot 138$
	$= -7(-1\cdot 138 + 216) + 4\cdot 138$	$= 11\cdot 138 - 7\cdot 216$
	$= 11(-14\cdot 216 + 3162) - 7\cdot 216$	$= -161\cdot 216 + 11\cdot 3162$

EDC Learning transforms lives.

The Squeeze

$\gcd(a, b)$	$=$	$= sa + tb$
6	$= -3 \cdot 18 + 60$	$= -3 \cdot 18 + 60$
	$= -3(-1 \cdot 60 + 78) + 60$	$= 4 \cdot 60 - 3 \cdot 78$
	$= 4(-1 \cdot 78 + 138) - 3 \cdot 78$	$= -7 \cdot 78 + 4 \cdot 138$
	$= -7(-1 \cdot 138 + 216) + 4 \cdot 138$	$= 11 \cdot 138 - 7 \cdot 216$
	$= 11(-14 \cdot 216 + 3162) - 7 \cdot 216$	$= -161 \cdot 216 + 11 \cdot 3162$

So,

$$t(a, b) = s\,(\mathrm{mod}(b, a), a)$$

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The Squeeze

$\gcd(a,b)$	$=$	$= sa + tb$
6	$= -3 \cdot 18 + 60$	$= -3 \cdot 18 + 60$
	$= -3(-1 \cdot 60 + 78) + 60$	$= 4 \cdot 60 - 3 \cdot 78$
	$= 4(-1 \cdot 78 + 138) - 3 \cdot 78$	$= -7 \cdot 78 + 4 \cdot 138$
	$= -7(-1 \cdot 138 + 216) + 4 \cdot 138$	$= 11 \cdot 138 - 7 \cdot 216$
	$= 11(-14 \cdot 216 + 3162) - 7 \cdot 216$	$= -161 \cdot 216 + 11 \cdot 3162$

So,

$$s(a,b) = t\,(\text{mod}(b,a),a) - \text{quot}(b,a) \cdot s\,(\text{mod}(b,a),a)$$

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The Squeeze

And, after lots of squinting, we have:

$$t(a,b) = \begin{cases} 1 & a = 0 \\ s\left(\text{mod}(b,a),a\right) & \text{otherwise} \end{cases}$$

$$s(a,b) = \begin{cases} 0 & a = 0 \\ t\left(\text{mod}(b,a),a\right) - \text{quot}(b,a) \cdot s\left(\text{mod}(b,a),a\right) & \text{otherwise} \end{cases}$$

Inverses

Suppose that a and b are relatively prime. Then

- $g(a, b) = 1$.
- Hence $s(a, b) \cdot a + t(a, b) \cdot b = 1$, and
- the remainder when $s(a, b) \cdot a$ is divided by b is 1.

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The Guts of Problem 1

I'm thinking of a number.
When I divide it by 3, the remainder is 2.
When I divide it by 5, the remainder is 3.
When I divide it by 7, the remainder is 1. What's my number?
What's the remainder when this beast:

$$2 \cdot (5 \cdot 7) \cdot s(5 \cdot 7, 3) + 3 \cdot (3 \cdot 7) \cdot s(3 \cdot 7, 5) + 1 \cdot (3 \cdot 5) \cdot s(3 \cdot 5, 7)$$

- is divided by 3? (it's 2)
- is divided by 5? (it's 3)
- is divided by 7? (it's 1)

This is the celebrated Chinese Remainder Theorem:

$$crt(a, b, c, m, n, p) = a \cdot np \cdot s(np, m) + b \cdot mp \cdot s(mp, n) + c \cdot mn \cdot s(mn, p)$$

Two Problems from *CME Algebra 2*

5. At Sasha's party, Tony presents the following puzzle: "I'm thinking of a number. If I divide it by 3, the remainder is 2. If I divide it by 5, the remainder is 3. If I divide it by 7, the remainder is 1. What's my number?"

a. What number might Tony be thinking of?

b. Is there more than one integer that fits Tony's puzzle? If so, name two of them. If not, explain why.

6. Later that night, Derman takes the floor and presents the following puzzle: "I'm thinking of a polynomial. If I divide it by $x - 3$, the remainder is 16. If I divide it by $x - 5$, the remainder is 42. If I divide it by $x - 7$, the remainder is 84. What's my polynomial?"

a. What polynomial might Derman be thinking of?

b. Is there more than one polynomial that fits Derman's puzzle? If so, name two of them. If not, explain why.

from Lesson 2.9, "Polynomial Division," page 153

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The Remainder Theorem (A-APR.2)

Theorem: If the polynomial $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$.

I'm thinking of a polynomial.

- I divide it by $x - 3$ and the remainder is 16. $f(3) = 16$.
- I divide it by $x - 5$ and the remainder is 42. $f(5) = 42$.
- I divide it by $x - 7$ and the remainder is 84. $f(7) = 84$.

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IT ALL GOES THROUGH

There's a Euclid algorithm, and hence a Chinese Remainder Theorem for polynomials, with one glitch.

$$\begin{array}{r} 3 \\ 2x^2 - x - 1 \overline{\smash{\big)}\, 6x^2 + x - 1} \\ \underline{6x^2 - 3x - 3} \\ 4x + 2 \end{array}$$

$$\begin{array}{r} \frac{1}{2}x - \frac{1}{2} \\ 4x + 2 \overline{\smash{\big)}\, 2x^2 - x - 1} \\ \underline{2x^2 - x - 1} \\ 0 \end{array}$$

Problem 2

Find a polynomial of smallest degree that agrees with this table:

Input	Output
3	16
5	42
7	84

We want

$$crt(16, 42, 84, x-3, x-5, x-7)$$

This is the celebrated Lagrange Interpolation Method

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SOME CONCLUSIONS

- General results often emerge from careful analysis of calculations.
  - CRT
- Structural similarities often emerge from careful analysis of calculations.
  - Lagrange Interpolation
- Careful analysis of calculations often requires intense concentration.
  - The s and t functions

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Some Conclusions

- Careful analysis of calculations is one core aspect of (algebraic) mathematical practice that integrates many of the standards.
  - The notion of *Euclidean ring*

- Certain uses of technology can help one articulate insights in precise mathematical language.
  - Shoehorning what you “feel” into mathematical language

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THANKS

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