Abstract Algebra for High School Teachers

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Outline

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ONE SIZE FITS ALL

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- What is this standard course? It usually has three parts: number theory, group theory, and commutative rings, all with equal attention.
- It wasn't always like that.

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- The syllabus is not connected very well to the high school teaching profession.

One Approach to a Solution

Our approach is to have a variant of the standard course that is not a watered down version of it, but one which is internally better organized—hence more interesting—and more applicable to what actually is done in high schools.

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The CBMS Recommendations

From MET-II: *. . . the mathematical topics in courses for prospective high school teachers and in professional development for practicing teachers should be tailored to the work of teaching.*

Learning Modern Algebra
from early attempts to prove Fermat's Last Theorem

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Learning Modern Algebra

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The standard course in abstract algebra isn't very useful to prospective teachers; it could be if

- it concentrated on commutative rings and fields over groups (MET-II),
- it motivated abstractions and proofs with concrete examples,
- it took a historical perspective,
- it made explicit connections to the mathematics of high school teaching,
- it was based on the actual work of high school teaching.

LEARNING MODERN ALGEBRA
FROM EARLY ATTEMPTS TO PROVE FERMAT'S LAST THEOREM

Some features of the course:

Learning Modern Algebra

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Some features of the course:

- Abstractions are motivated with examples whenever possible.
- Groups are introduced in the last chapter, as part of an introduction to Galois theory.
- The structural similarities between between $\mathbb{Z}$ and $k[x]$ are made central throughout.
- The development follows the historical evolution of the ideas.
- It develops topics that are foundational for high school teaching
  - in "Connections" sections, and
  - integrated throughout.

Learning Modern Algebra

from early attempts to prove Fermat's Last Theorem

A sample:

- Pythagorean triples and "The Method of Diophantus."
- A method for teaching mathematical induction.
- Fibonacci numbers.
- A historical development of $\mathbb{C}$.
- The mathematics of task design.
- Periods of repeating decimals.

Learning Modern Algebra
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A sample, cont'd

- Cryptography.
- Lagrange interpolation and the CRT.
- Ruler and compass constructions.
- Gauss' construction of the regular 17-gon.
- The arithmetic of $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$.
- Solvability by radicals.
- FLT for exponents 3, 4, and certain n.

Learning Modern Algebra

from Early Attempts to Prove Fermat's Last Theorem

Current organization:

1. Early Number Theory
2. Induction
3. Renaissance
4. Modular Arithmetic
5. Abstract Algebra
6. Arithmetic with Polynomials
7. Quotients, Fields, and Classical Problems
8. Cyclotomic Integers
9. Epilog

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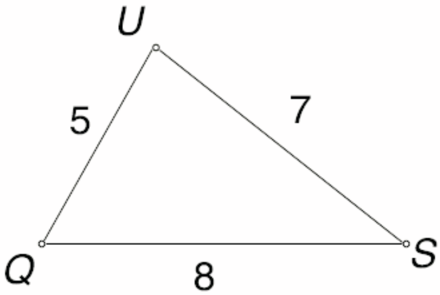
Example: The Mathematics of Task Design

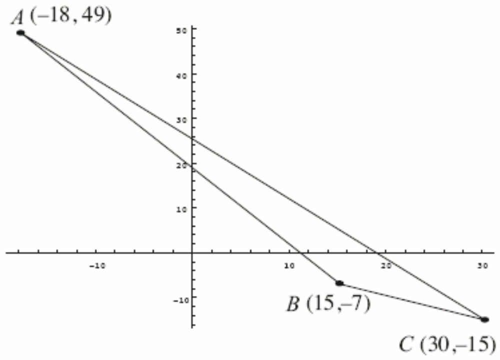


How big is angle *Q*?

The Mathematics of Task Design

The vertices of a triangle have coordinates

$$(-18, 49), (15, -7), (30, -15)$$



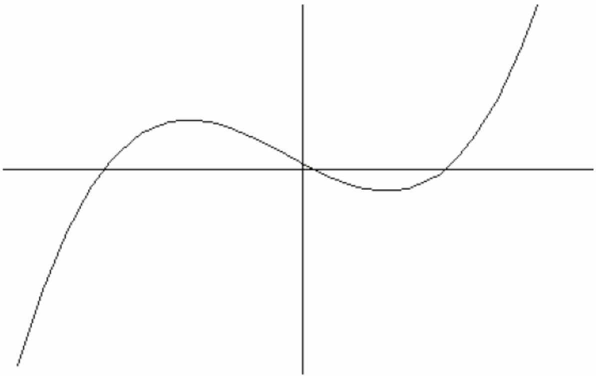
A strange but nice triangle; how long are the sides?

THE MATHEMATICS OF TASK DESIGN

Fold up to make a box

What size cut-out maximizes the volume?

The Mathematics of Task Design

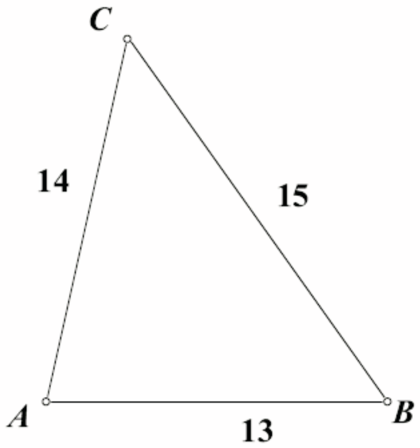


$f(x) = 140 - 144\,x + 3\,x^2 + x^3$

Find the zeros, extrema, and inflection points

The Mathematics of Task Design

Find the area of this triangle



An Algebraic Approach
How to Amaze Your friends at Parties

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$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$$

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Pick your favorite Gaussian Integer (make $a > b > 0$) and square it.

Why Does this Work?
It's "in there"

- If $z = a + bi$ is a Gaussian integer, the complex conjugate of z is $\overline{z} = a - bi$
- The following properties of conjugation hold:
  - $\overline{z + w} = \overline{z} + \overline{w}$
  - $\overline{zw} = \overline{z}\,\overline{w}$
  - $z\,\overline{z} = a^2 + b^2$, a non-negative integer.
- The norm of z, $N(z)$, is defined by $N(z) = z\,\overline{z}$.

Why Does this Work?
It's "in there"

- Norm has the following properties:
  - $N(zw) = N(z)\,N(w)$ for all Gaussian integers z and w.
  - Hence, if z is a Gaussian integer, then
  $$N(z^2) = (N(z))^2$$
  - If $z = a + bi$, $N(z) = a^2 + b^2$, a non-negative integer.

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  - If $z = a + bi$, $N(z) = a^2 + b^2$, a non-negative integer.

So, to make $N(z)$ a square in $\mathbb{Z}$, make z a square in $\mathbb{Z}[i]$.

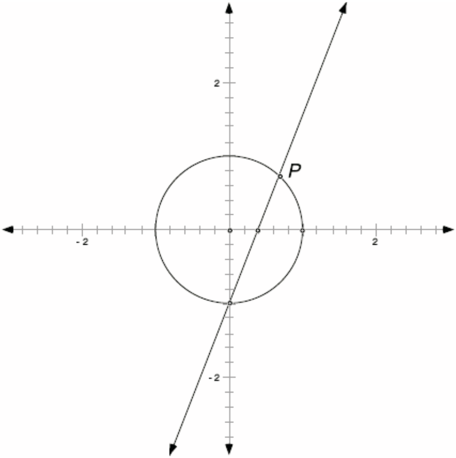
An Algebraic Approach

$s \rightarrow$ / $r \downarrow$	1	2	3
2	$3+4i, 5$		
3	$8+6i, 10$	$5+12i, 13$	
4	$15+8i, 17$	$12+16i, 20$	$7+24i, 25$
5	$24+10i, 26$	$21+20i, 29$	$16+30i, 34$
6	$35+12i, 37$	$32+24i, 40$	$27+36i, 45$
7	$48+14i, 50$	$45+28i, 53$	$40+42i, 58$
8	$63+16i, 65$	$60+32i, 68$	$55+48i, 73$

An Algebraic Approach

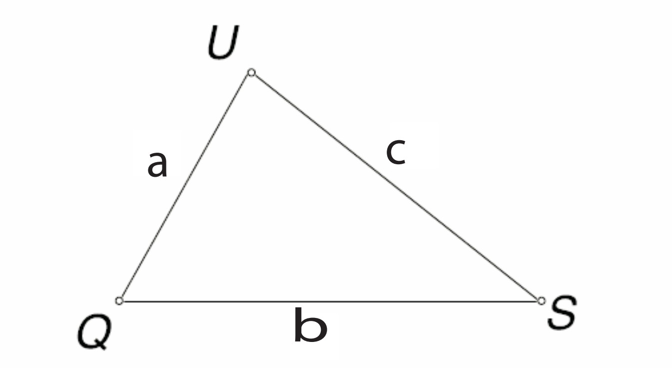
$s \rightarrow$ / $r \downarrow$	4	5	6
5	$9+40i, 41$		
6	$20+48i, 52$	$11+60i, 61$	
7	$33+56i, 65$	$24+70i, 74$	$13+84i, 85$
8	$48+64i, 80$	$39+80i, 89$	$28+96i, 100$

A GEOMETRIC APPROACH
AFTER DICK ASKEY



If the line has rational slope, *P* has rational coordinates

The Mathematics of Task Design



A nice triangle

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos 60^\circ \\ &= a^2 + b^2 - 2ab\,\tfrac{1}{2} \\ &= a^2 - ab + b^2 \end{aligned}$$

An Algebraic Approach

So, we want integers (a, b, c) so that

$$c^2 = a^2 - ab + b^2$$

Call such a triple an "Eisenstein Triple."
Let

$$\omega = \frac{-1 + i\sqrt{3}}{2} \quad \text{so that} \quad \omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0$$

and consider $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$.

Then $(a + b\omega)(a + b\overline{\omega}) = a^2 - ab + b^2$.

An Algebraic Approach

Start with $z = 3 + 2\omega$, so that $N(z) = 7$. Square it:

$$\begin{aligned} z^2 &= (3+2\omega)^2 \\ &= 9 + 12\omega + 4\omega^2 \\ &= 9 + 12\omega + 4(-1-\omega) \quad (\omega^2 + \omega + 1 = 0) \\ &= 5 + 8\omega \end{aligned}$$

and voilà:

$$5^2 - 5 \cdot 8 + 8^2 = N(z^2) = (N(z))^2 = 49, \quad \text{a perfect square.}$$

So the triangle whose sides have length 5, 8, and 7 has a 60° angle.

An Algebraic Approach

$s \rightarrow$ / r	1	2	3	4
2	$3+3\omega, 3$			
3	$8+5\omega, 7$	$5+8\omega, 7$		
4	$15+7\omega, 13$	$12+12\omega, 12$	$7+15\omega, 13$	
5	$24+9\omega, 21$	$21+16\omega, 19$	$16+21\omega, 19$	$9+24\omega, 21$
6	$35+11\omega, 31$	$32+20\omega, 28$	$27+27\omega, 27$	$20+32\omega, 28$
7	$48+13\omega, 43$	$45+24\omega, 39$	$40+33\omega, 37$	$33+40\omega, 37$
8	$63+15\omega, 57$	$60+28\omega, 52$	$55+39\omega, 49$	$48+48\omega, 48$
9	$80+17\omega, 73$	$77+32\omega, 67$	$72+45\omega, 63$	$65+56\omega, 61$
10	$99+19\omega, 91$	$96+36\omega, 84$	$91+51\omega, 79$	$84+64\omega, 76$

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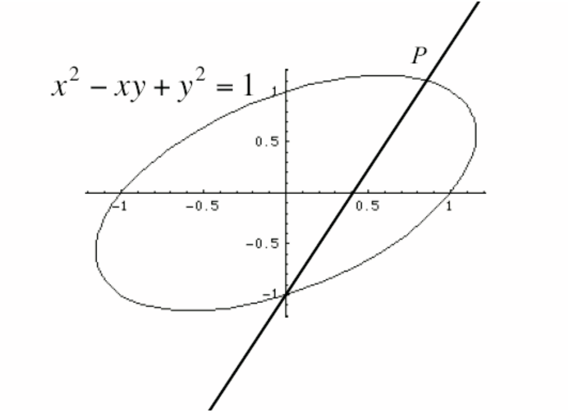

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CONCLUSIONS

- Mathematics teaching, like any mathematical profession, involves profession-specific applications of classical mathematics.

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- The mathematics involved in high school and in the teaching high of high school can lead one into deep mathematical results and methods.

AND FINALLY

- The kind of mathematics used in mathematics teaching is a valuable arena of applications for *all* mathematics majors.

Thanks

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Slides (soon) available at

cmeproject.edc.org