Knowledge of Mathematics for Teaching High School

Suggestions for mathematical preparation and professional development for teachers

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Outline

Knowing Mathematics

Claim: The common mantra

"Mathematics teachers need to know the content in order to be effective"

is too vague to be useful.

- A finer analysis of content knowledge, one that considers the demands of the profession, is needed.
- Such an analysis comes from a close look at how expert teachers "know" mathematics and use it in their work.

LOOK AT EXPERT TEACHERS

Expert teachers know mathematics as a scholar:

They have a solid grounding in classical mathematics, including

- its major results
- its history of ideas
- its connections to precollege mathematics

Look at Expert Teachers

Expert teachers know mathematics as an educator:

They understand the habits of mind that underlie major branches of mathematics and how they develop in learners, including

- algebra and arithmetic
- geometry
- analysis

Look at Expert Teachers

Expert teachers know mathematics as a mathematician:

They have *experienced the doing of mathematics*—they know, in a very personal way, what it's like to

- grapple with problems
- build abstractions
- develop theories
- become completely absorbed in mathematical activity for a sustained period of time

Look at Expert Teachers

Expert teachers know mathematics as a teacher:

They are expert in uses of mathematics that are specific to the profession, including

- the ability “to think deeply about simple things” (*Arnold Ross*)
- the craft of task design
- the understanding of logical sequences in the syllabus
- the “mining” of student ideas

Look at Expert Teachers

Mathematics for teaching demands "knowing":

- The "facts"
- The "epistemology"
- The "experience"
- The "craft"

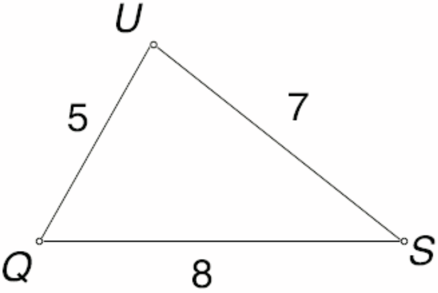
Look at Expert Teachers

Mathematics for teaching demands “knowing”:

- The “facts”
- The “epistemology”
- The “experience” $\leftarrow$
- The “craft” $\leftarrow$

The Mathematics of Task Design

Getting Started

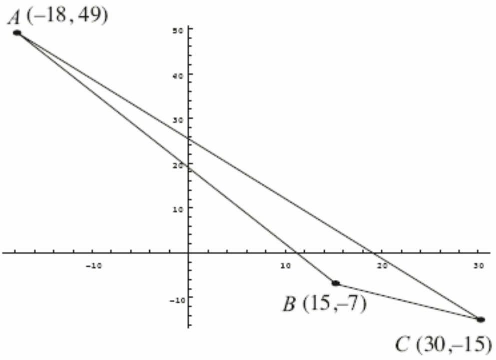


How big is angle Q?

The Mathematics of Task Design

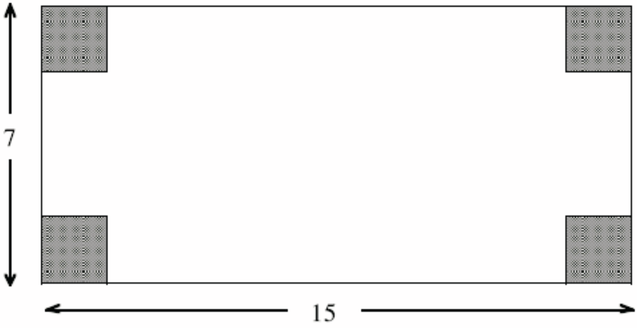
The vertices of a triangle have coordinates

$$(-18, 49), (15, -7), (30, -15)$$



A strange but nice triangle; how long are the sides?

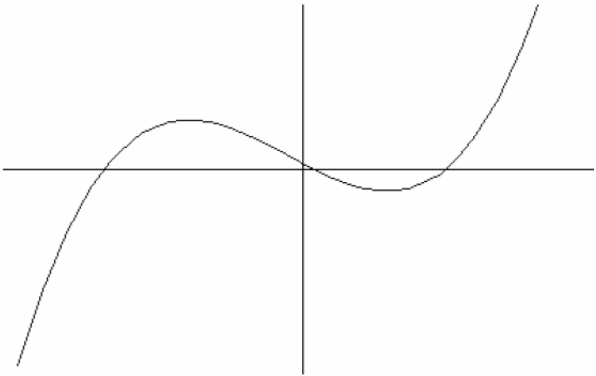
The Mathematics of Task Design



Fold up to make a box

What size cut-out maximizes the volume?

The Mathematics of Task Design

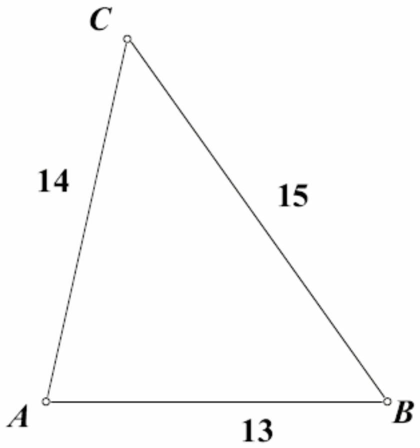


$f(x) = 140 - 144\,x + 3\,x^2 + x^3$

Find the zeros, extrema, and inflection points

The Mathematics of Task Design

Find the area of this triangle



An Algebraic Approach

How to Amaze Your friends at Parties

The *Gaussian integers* is the ring

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$$

Pick your favorite Gaussian Integer (make $a > b > 0$) and square it.

Why Does this Work?

It's "in there"

- If $z = a + bi$ is a Gaussian integer, the complex conjugate of z is $\overline{z} = a - bi$
- The following properties of conjugation hold:
  - $\overline{z + w} = \overline{z} + \overline{w}$
  - $\overline{zw} = \overline{z}\,\overline{w}$
  - $z\,\overline{z} = a^2 + b^2$, a non-negative integer.
- The norm of z, $N(z)$, is defined by $N(z) = z\,\overline{z}$.

Why Does this Work?

It's "in there"

- Norm has the following properties:
  - $N(zw) = N(z)\,N(w)$ for all Gaussian integers z and w.
  - Hence, if z is a Gaussian integer, then

$$N(z^2) = (N(z))^2$$

  - If $z = a + bi$, $N(z) = a^2 + b^2$, a non-negative integer.

So, to make $N(z)$ a square in $\mathbb{Z}$, make z a square in $\mathbb{Z}[i]$.

An Algebraic Approach

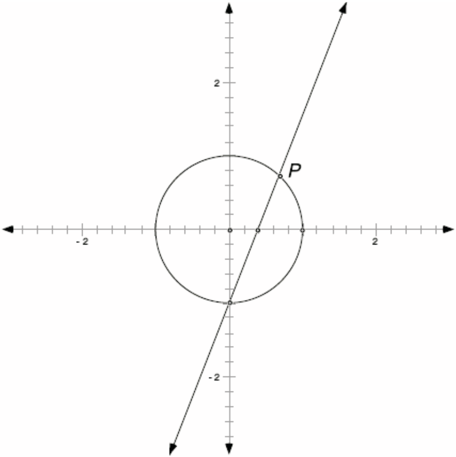
$s \rightarrow$ / $r \downarrow$	1	2	3
2	$3+4i, 5$		
3	$8+6i, 10$	$5+12i, 13$	
4	$15+8i, 17$	$12+16i, 20$	$7+24i, 25$
5	$24+10i, 26$	$21+20i, 29$	$16+30i, 34$
6	$35+12i, 37$	$32+24i, 40$	$27+36i, 45$
7	$48+14i, 50$	$45+28i, 53$	$40+42i, 58$
8	$63+16i, 65$	$60+32i, 68$	$55+48i, 73$

An Algebraic Approach

$s \rightarrow$ $r \downarrow$	4	5	6
5	$9 + 40i, 41$		
6	$20 + 48i, 52$	$11 + 60i, 61$	
7	$33 + 56i, 65$	$24 + 70i, 74$	$13 + 84i, 85$
8	$48 + 64i, 80$	$39 + 80i, 89$	$28 + 96i, 100$

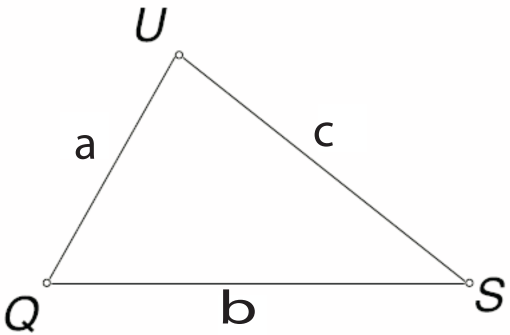
A Geometric Approach

After Dick Askey



If the line has rational slope, *P* has rational coordinates

THE MATHEMATICS OF TASK DESIGN



A nice triangle

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos 60^\circ \\
&= a^2 + b^2 - 2ab\,\tfrac{1}{2} \\
&= a^2 - ab + b^2
\end{aligned}
$$

An Algebraic Approach

So, we want integers (a, b, c) so that

$$c^2 = a^2 - ab + b^2$$

Call such a triple an "Eisenstein Triple."
Let

$$\omega = \frac{-1 + i\sqrt{3}}{2} \quad \text{so that} \quad \omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0$$

and consider $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$.

Then $(a + b\omega)(a + b\overline{\omega}) = a^2 - ab + b^2$.

An Algebraic Approach

Start with $z = 3 + 2\omega$, so that $N(z) = 7$. Square it:

$$\begin{aligned} z^2 &= (3 + 2\omega)^2 \\ &= 9 + 12\omega + 4\omega^2 \\ &= 9 + 12\omega + 4(-1 - \omega) \quad (\omega^2 + \omega + 1 = 0) \\ &= 5 + 8\omega \end{aligned}$$

and voilà:

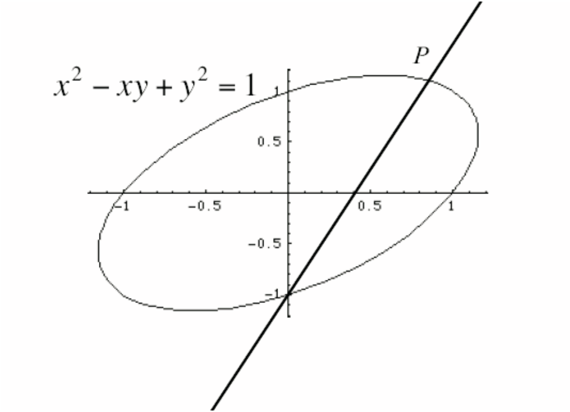
$$5^2 - 5 \cdot 8 + 8^2 = N(z^2) = (N(z))^2 = 49, \quad \text{a perfect square.}$$

So the triangle whose sides have length 5, 8, and 7 has a 60° angle.

An Algebraic Approach

$s \rightarrow$ r	1	2	3	4
2	$3 + 3\omega, 3$			
3	$8 + 5\omega, 7$	$5 + 8\omega, 7$		
4	$15 + 7\omega, 13$	$12 + 12\omega, 12$	$7 + 15\omega, 13$	
5	$24 + 9\omega, 21$	$21 + 16\omega, 19$	$16 + 21\omega, 19$	$9 + 24\omega, 21$
6	$35 + 11\omega, 31$	$32 + 20\omega, 28$	$27 + 27\omega, 27$	$20 + 32\omega, 28$
7	$48 + 13\omega, 43$	$45 + 24\omega, 39$	$40 + 33\omega, 37$	$33 + 40\omega, 37$
8	$63 + 15\omega, 57$	$60 + 28\omega, 52$	$55 + 39\omega, 49$	$48 + 48\omega, 48$
9	$80 + 17\omega, 73$	$77 + 32\omega, 67$	$72 + 45\omega, 63$	$65 + 56\omega, 61$
10	$99 + 19\omega, 91$	$96 + 36\omega, 84$	$91 + 51\omega, 79$	$84 + 64\omega, 76$

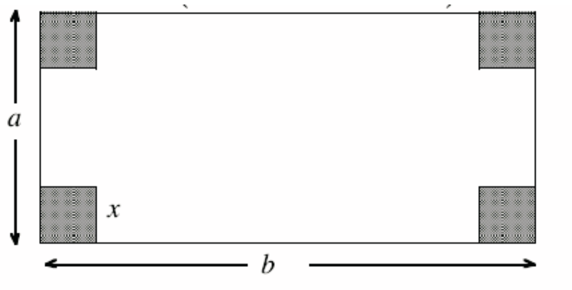
A Geometric Approach



If the line has rational slope, *P* has rational coordinates

Other Examples

Reasonable and Rational Boxes



Soon to be a box

Well, as we tell our students, let the size of the cut-out be x. Then the volume is a function of x:

$$V(x) = (a - 2x)(b - 2x)x = 4x^3 - 2(a + b)x^2 + abx,$$

Other Examples

Reasonable and Rational Boxes

$$V(x) = 4x^3 - 2(a+b)x^2 + abx,$$

so,

$$V'(x) = 12x^2 - 4(a+b)x + ab.$$

We want this to have rational zeros, so we want the discriminant

$$16(a+b)^2 - 48ab$$

to be a perfect square. But 16 is a perfect square, so we want to make

$$(a+b)^2 - 3ab = a^2 - ab + b^2$$

a perfect square. We can do this by taking a and b to be the legs of an Eisenstein triple.

Other Examples
Clean Cubics

We (all of us) want cubic polynomials

$$f(x) = ax^3 + bx^2 + cx + d$$

with integer coefficients, zeros, extrema, and inflection points.

With a little more work, Eisenstein integers can be used here, too.

Clean Cubics

By a some classical theory of equations—scaling and "completing the square"—we can change variables and assume that the cubic f is reduced—it has form

$$f(x) = x^3 + qx + r$$

This immediately guarantees that $f''(x) = 24x$ has an integer root, namely, 0 (the inflection point of the graph is on the y-axis).

Next, if we replace q by $-3p^2$ for some integer p, then $f'(x) = 3x^2 - 3p^2$, and $f'(x)$ also has integer roots.

Clean Cubics

So, our cubic now looks like $f(x) = x^3 - 3p^2 x + r$. This will have rational extrema and inflection points, so all we have to do is ensure that it has three rational roots.

If f has two rational roots, it has three. So it's enough to make two roots, say, $-\alpha$ and β ($-\alpha \neq \beta$), rational.

But if $f(-\alpha) = f(\beta) = 0$, we have

$$-\alpha^3 + 3p^2\alpha = \beta^3 - 3p^2\beta$$

or

$$\beta^3 + \alpha^3 = 3p^2(\alpha + \beta).$$

Divide both sides by $\alpha + \beta$ to obtain

$$\alpha^2 - \alpha\beta + \beta^2 = 3p^2.$$

Clean Cubics

$$\alpha^2 - \alpha\beta + \beta^2 = 3p^2.$$

Eisenstein integers again. This is the same as

$$N(\alpha + \beta\omega) = 3p^2.$$

This time we want an Eisenstein integer whose norm is 3 times a square.

Clean Cubics

We're in luck: $N(1-\omega) = 3$

Hence we just need to take $\alpha + \beta\omega$ to be $1-\omega$ times the square of an Eisenstein integer. Indeed, if

$$\alpha + \beta\omega = (1-\omega)z^2 \quad (z \in \mathbb{Z}[\omega])$$

then

$$\begin{aligned} \alpha^2 - \alpha\beta + \beta^2 = N(\alpha + \beta\omega) &= N\left((1-\omega)z^2\right) \\ &= N(1-\omega)\, N\left(z^2\right) \\ &= 3\,(N(z))^2\,, \end{aligned}$$

which is 3 times the square of an integer.

Clean Cubics

So, if $\alpha + \beta\omega = (1 - \omega)z^2$, then

1. $N(\alpha + \beta\omega) = 3p^2$ for some integer p,
2. $-\alpha$ and β are roots of a cubic equation

$$x^3 - 3p^2x + (\alpha^3 - 3p^2\alpha) = 0$$

And this gives an algorithm for generating our cubics:

Clean Cubics

$s \rightarrow$ / $r \downarrow$	1	2	3
2	$54 - 27x + x^3$	$-128 - 48x + x^3$	
3	$286 - 147x + x^3$	$286 - 147x + x^3$	$-1458 - 243x + x^3$
4	$-506 - 507x + x^3$	$3456 - 432x + x^3$	$-506 - 507x + x^3$
5	$-7722 - 1323x + x^3$	$10582 - 1083x + x^3$	$10582 - 1083x + x^3$
6	$-35282 - 2883x + x^3$	$18304 - 2352x + x^3$	$39366 - 2187x + x^3$

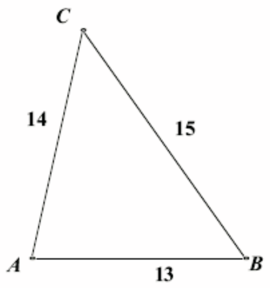
Here,

$$\begin{aligned}(1-\omega)(r+s\omega)^2 &= \alpha + \beta\omega \\ 3p^2 &= N(\alpha+\beta\omega) \\ d &= \alpha^3 - 3p^2\alpha\end{aligned}$$

and $f(x) = x^3 - 3p^2x + d$.

These can be translated to produce examples with a non-zero x^2 term.

Other Examples
Heron Triangles

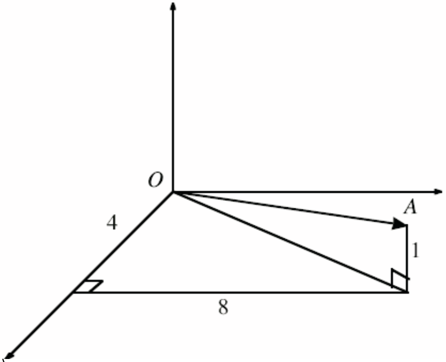


A $(13, 14, 15)$ triangle

$$A = \frac{1}{4}\sqrt{(13+14+15)(13+14-15)(13+15-14)(14+15-13)}$$
$$= 84$$

Other Examples
Vectors in $\mathbb{Z}^3$

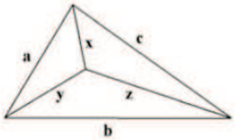
Find vectors in $\mathbb{Z}^3$ with integral length.
Ex: $(4, 8, 1)$



How long is $\overrightarrow{OA}$?

Other Examples
Nice Fermat triangles

Find *Matsuura Triples*: integer sided triangles so that the Fermat point is an integer distance from each vertex.

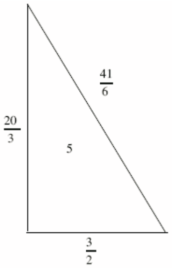


x	y	z	a	b	c
195	264	325	399	511	455
264	325	440	511	665	616
390	528	650	798	1022	910
528	650	880	1022	1330	1232
585	792	975	1197	1533	1365

Nice Fermat Triangles

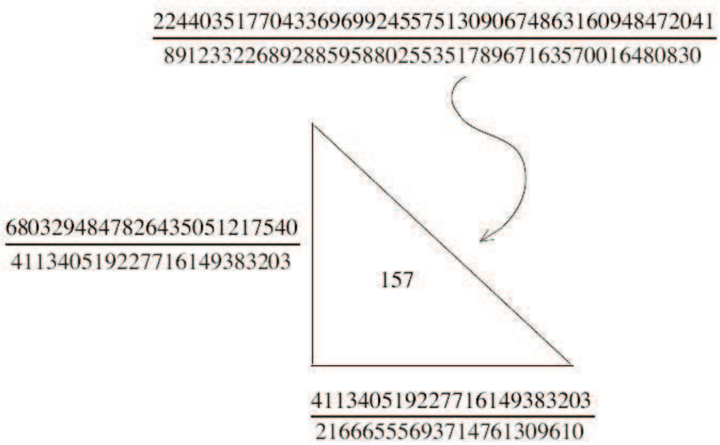
Other Examples
Congruent numbers

Find *congruent numbers*: Integers that are areas of right triangles with rational side lengths.
Example: 6 is the area of a $(3, 4, 5)$ right triangle.
Non-example: 1 is not a congruent number (Fermat).
Example: 5 is a congruent number.



Example: 157 is a congruent number.

Other Examples
Congruent Numbers



157 is a congruent number.

High School Teaching

A Mathematical Profession

Topics: The mathematics in courses for prospective high school teachers should be tailored to the work of teaching, examining connections between middle grades and high school mathematics as well as those between high school and college.

Practice: Teachers need opportunities for the full range of mathematical experience themselves: struggling with hard problems, discovering their own solutions, reasoning mathematically, modeling with mathematics, and developing mathematical habits of mind.

Scope: It is impossible to learn all the mathematics one will use in any mathematical profession, including teaching, in four years of college—teachers need opportunities to learn further mathematics throughout their careers.

TOPICS

With regard to topics, teachers might take standard courses for mathematics majors. MET-II describes ways in which such courses can be adjusted to better connect with the mathematics of high school.

Here “the mathematics of high school” does not mean simply the syllabus of high school mathematics, the list of topics in a typical high school text. Rather it is the structure of mathematical ideas from which that syllabus is derived.

Practice
Emphasis on reasoning

A primary goal of a mathematics major program is the development of mathematical reasoning skills.

This may seem like a truism to mathematics faculty, to whom reasoning is second nature. But precisely because it is second nature, it is often not made explicit in undergraduate mathematics courses.

A mathematician may use reasoning by continuity to come to a conjecture, but what college students see is often the end result of this thinking, with no idea about how it was conceived.

Mathematical Practice

It is possible that the discontinuity between how mathematicians and teachers view the whole enterprise of mathematics—what is important, what is convention, what constitutes expertise, and even what it means to understand the subject—is because the typical mathematics major does not provide an intense immersion experience in mathematics.

The research experiences available in many departments and summer programs are recommended for prospective mathematics teachers.

Scope: Undergraduate Preparation

The mathematics courses taken by prospective high teachers include at least

- a three-course calculus sequence,
- an introductory statistics course,
- an introductory linear algebra course, and
- 18 additional semester-hours of advanced mathematics, *including 9 semester-hours explicitly focused on high school mathematics from an advanced standpoint*.

It's desirable to have a further 9 semester-hours of mathematics.

Scope: Professional Development

Many teachers prepared before the era of the CCSS will need opportunities to study content that they have not previously taught, particularly in the areas of statistics and probability.

In addition to learning more mathematical topics, teachers need experiences that renew and strengthen their interest in and love for mathematics, help them represent mathematics as a living discipline to their students by exemplifying mathematical practices, and figure out how to pose tasks to students that highlight the essential ideas under consideration.

Scope: Professional Development

A Role for Mathematicians

The involvement of the mathematical community in career-long, content-based professional development programs for practicing teachers provides an opportunity for mathematicians and statisticians to have a profound effect on the content and direction of high school mathematics. And it provides teachers with years of opportunities to learn more mathematics and statistics that is especially useful in their profession and to be partners with mathematicians and statisticians in a desperately needed effort to improve professional development experiences.

Learning Modern Algebra

from early attempts to prove Fermat's Last Theorem

The standard course in abstract algebra isn't very useful to prospective teachers; it could be if

- it concentrated on commutative rings and fields (over groups),
- it motivated abstractions and proofs with concrete examples,
- it took a historical perspective,
- it made explicit connections to the mathematics of high school teaching,
- it was based on the actual work of high school teaching.

Learning Modern Algebra

from early attempts to prove Fermat's Last Theorem

Some features of the course:

- Abstractions are motivated with examples whenever possible.
- Groups are introduced in the last chapter, as part of an introduction to Galois theory.
- The structural similarities between between $\mathbb{Z}$ and $k[x]$ are made central throughout.
- The development follows the historical evolution of the ideas.
- It develops topics that are foundational for high school teaching
  - in "Connections" sections, and
  - integrated throughout.

Learning Modern Algebra

from early attempts to prove Fermat's Last Theorem

A sample:

- Pythagorean triples and "The Method of Diophantus."
- A method for teaching mathematical induction.
- A historical development of $\mathbb{C}$.
- The mathematics of task design.
- Periods of repeating decimals.
- Cryptography.
- Lagrange interpolation and the CRT.
- Ruler and compass constructions.
- Gauss' construction of the regular 17-gon.
- The arithmetic of $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$.
- Solvability by radicals.
- FLT for exponents 3, 4, and certain n.

Learning Modern Algebra

from early attempts to prove Fermat's Last Theorem

Current organization:

1. Early Number Theory
2. Induction
3. Renaissance
4. Modular Arithmetic
5. Abstract Algebra
6. Arithmetic with Polynomials
7. Quotients, Fields, and Classical Problems
8. An Introduction to Algebraic Number Theory
9. Epilog

CONCLUSIONS

- Mathematics teaching, like any mathematical profession, involves profession-specific applications of classical mathematics.
- The mathematics involved in high school and in the teaching high of school can lead one into deep mathematical results and methods.
- Mathematicians are ideally suited to the work of teasing out these results and methods.

  This is exciting work that engages mathematicians in new ideas and practices that are important to them **as mathematicians***.*

—Glenn Stevens

AND FINALLY

- The kind of mathematics used in mathematics teaching is a valuable arena of applications for *all* mathematics majors.

REFERENCES:

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[2] A. Cuoco, “Meta-Problems in Mathematics,” *College Mathematics Journal*, November, 2000.

[3] A. Cuoco and J. Rotman, *Learning Modern Algebra From Early Attempts to Prove Fermat’s Last Theorem*, MAA, Forthcoming in 2013.

Thanks

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Slides (soon) available at

cmeproject.edc.org