Reasoning and Making Sense of Algebra

The Standards for Mathematical Practice in Grades 9–12

Al Cuoco
Center for Mathematics Education
Education Development Center, Inc.

KCM 2012 Conference

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   - Caveat

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The Notion of Mathematical Practice

What mathematicians most wanted and needed from me was to learn my ways of thinking, and not in fact to learn my proof of the geometrization conjecture for Haken manifolds.

—William Thurston
"On Proof and Progress in Mathematics."
Bulletin of the American Mathematical Society, 1994

The Notion of Mathematical Practice

The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more importantly, from developing the ways of thinking—the habits of mind—used to create the results.

—Al Cuoco, Paul Goldenberg, & June Mark
"Habits of Mind: An Organizing Principle for High School Curricula." *The Journal of Mathematical Behavior*, 1996.

The Notion of Mathematical Practice

It will be helpful to name and (at least partially) specify some of the things—practices, dispositions, sensibilities, habits of mind—entailed in doing mathematics. . . . These are things that mathematicians typically do when they do mathematics. At the same time most of these things, suitably interpreted or adapted, could apply usefully to elementary mathematics no less than to research.

—Hyman Bass

“A Vignette of Doing Mathematics.” *The Montana Mathematics Enthusiast*, 2011.

The Notion of Mathematical Practice

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise. . . . Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

—CCSS, 2010

From a Popular Text (~ 1980)

"Factoring Pattern for $x^2 + bx + c$, c Negative"

Factor. Check by multiplying factors. If the polynomial is not factorable, write "prime."

1. $a^2 + 4a - 5$	**2.** $x^2 - 2x - 3$	**3.** $y^2 - 5y - 6$
4. $b^2 + 2b - 15$	**5.** $c^2 - 11c - 10$	**6.** $r^2 - 16r - 28$
7. $x^2 - 6x - 18$	**8.** $y^2 - 10c - 24$	**9.** $a^2 + 2a - 35$
10. $k^2 - 2k - 20$	**11.** $z^2 + 5z - 36$	**12.** $r^2 - 3r - 40$
13. $p^2 - 4p - 21$	**14.** $a^2 + 3a - 54$	**15.** $y^2 - 5y - 30$
16. $z^2 - z - 72$	**17.** $a^2 - ab - 30b^2$	**18.** $k^2 - 11kd - 60d^2$
19. $p^2 - 5pq - 50q^2$	**20.** $a^2 - 4ab - 77b^2$	**21.** $y^2 - 2yz - 3z^2$
22. $s^2 + 14st - 72t^2$	**23.** $x^2 - 9xy - 22y^2$	**24.** $p^2 - pq - 72q^2$

From a Published Text (2010)

To factor a trinomial of the form $ax^2 + bx + c$ where $a > 0$, follow these steps:

A	B	C
F	H	D
G	I	E

1. Identify the values of a, b, c. Put a in Box A and c in Box B. Put the product of a and c in Box C.
2. List the factors of the number from Box C and identify the pair whose sum is b. Put the two factors you find in Box D and E.
3. Find the greatest common factor of Boxes A [sic] and E and put it in box G.

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From a Published Text (2010)

A	B	C
F	H	D
G	I	E

4. In Box F, place the number you multiply by Box G to get Box A.
5. In Box H, place the number you multiply by Box F to get Box D.
6. In Box I, place the number you multiply by Box G to get Box E.

Solution: The binomial factors whose product gives the trinomial are $(Fx + I)(Gx + H)$.

Seeing and Using Structure in Expressions

Factoring monic quadratics:

"Sum-Product" problems

$$x^2 + 14x + 48$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

so...

Find two numbers whose sum is 14 and whose product is 48.

$$(x + 6)(x + 8)$$

Seeing and Using Structure in Expressions

What about this one?

$$49x^2 + 35x + 6$$

$$\begin{aligned} 49x^2 + 35x + 6 &= (7x)^2 + 5(7x) + 6 \\ &= \clubsuit^2 + 5\clubsuit + 6 \\ &= (\clubsuit + 3)(\clubsuit + 2) \\ &= (7x + 3)(7x + 2) \end{aligned}$$

Seeing and Using Structure in Expressions

What about this one?

$$6x^2 + 31x + 35$$

$$\begin{aligned} 6(6x^2 + 31x + 35) &= (6x)^2 + 31(6x) + 210 \\ &= \clubsuit^2 + 31\clubsuit + 210 \\ &= (\clubsuit + 21)(\clubsuit + 10) \\ &= (6x + 21)(6x + 10) \\ &= 3(2x + 7) \cdot 2(3x + 5) \\ &= 6(2x + 7)(3x + 5) \quad \textit{so} \ldots \end{aligned}$$

$$6(6x^2 + 31x + 35) = 6(2x + 7)(3x + 5)$$

Seeing and Using Structure in Expressions

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$$\not{6}(6x^2 + 31x + 35) = \not{6}(2x + 7)(3x + 5)$$

EDC Learning transforms lives.

Seeing and Using Structure in Expressions

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$$6x^2 + 31x + 35 \;=\; (2x + 7)(3x + 5)$$

Seeing and Using Structure in Expressions

- This technique is perfectly general and can be used to transform a polynomial of any degree into one whose leading coefficient is 1.
- And it fits into the larger landscape of the *theory of equations* that shows how to use similar transformations to
  - remove terms
  - transform roots
  - derive “formulas” for equations of degree 3 and 4
  - extend the notion of *discriminant* to higher degrees

Other Examples Where Chunking Is Useful

- Completing the square and removing terms
- Solving trig equations
- Analyzing conics and other curves
- All over calculus
- Interpreting results from a computer algebra system

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Some Thorny Topics in Algebra

1. Students have trouble expressing generality with algebraic notation.
2. This is especially prevalent when they have to set up equations to solve word problems.
3. Many students have difficulty with slope, graphing lines, and finding equations of lines.
4. Building and using algebraic functions is another place where students struggle.

This list looks like a collection of disparate topics—using notation, solving word problems, ...

Some Thorny Topics in Algebra

But if one looks underneath the topics to the mathematical habits that would help students master them, one finds a remarkable similarity:

A key ingredient in such a mastery is the reasoning habit of seeking and expressing regularity in repeated calculations.

- This habit manifests itself when one is performing the same calculation over and over and begins to notice the “rhythm” in the operations.
- Articulating this regularity leads to a generic algorithm, typically expressed with algebraic symbolism, that can be applied to any instance and that can be transformed to reveal additional meaning, often leading to a solution of the problem at hand.

Example A: The Dreaded Algebra Word Problem

Think about how hard it is for students to set up an equation that can be used to solve an algebra word problem. Some reasons for the difficulties include reading levels and unfamiliar contexts. But there has to be more to it than these surface features.

Consider, for example, the following two problems.

Example A: The Dreaded Algebra Word Problem

1. The driving distance from Boston to Chicago is is 990 miles. Derman drives from Boston to Chicago at an average speed of 50mph and returns at an average speed of 60mph. For how many hours is Derman on the road?
2. Derman drives from Boston to Chicago at an average speed of 50mph and returns at an average speed of 60mph. Derman is on the road for 36 hours. What is the driving distance from Boston to Chicago?

The problems have identical reading levels, and the context is the same in each. But teachers report that many students who can solve problem 1 are baffled by problem 2.

Example A: The Dreaded Word Problem

This is where the reasoning habit of “expressing the rhythm” in a calculation can be of great use. The basic idea:

- Guess at an answer to problem 2, and
- check your guess as if you were working on problem 1, *keeping track of your steps*.

The purpose of the guess is not to stumble on (or to approximate) the correct answer; rather, it is to help you construct a “checking algorithm” that will work for any guess.

Example A: The Dreaded Word Problem

Problem 2: Derman drives over at an average speed of 50mph and returns at an average of 60mph. He's on the road for 36 hours. What is the driving distance?

So, you take several guesses until you are able to express your checking algorithm in algebraic symbols. For example:

- Suppose the distance is 1000 miles.
- How do I check the guess of 1000 miles? I divide 1000 by 50. Then I divide 1000 by 60. Then I add my answers together, to see if I get 36. I don't.
- So, check another number—say 950. 950 divided by 50 plus 950 divided by 60. Is that 36?
- No, but a general method is evolving that will allow me to check *any* guess.

Example A: The Dreaded Word Problem

- My guess-checker is

$$\frac{\text{guess}}{50} + \frac{\text{guess}}{60} \stackrel{?}{=} 36$$

- So my *equation* is

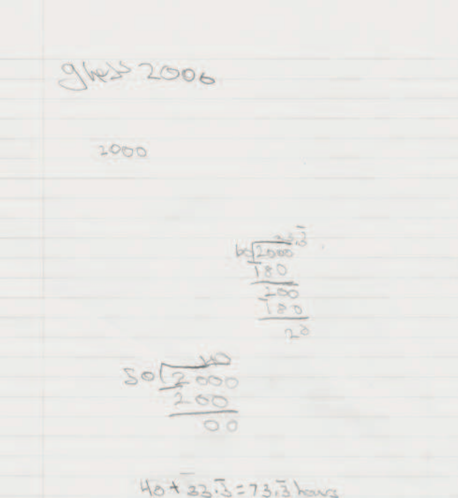
$$\frac{\text{guess}}{50} + \frac{\text{guess}}{60} = 36$$

or, letting x stand for the unknown correct guess,

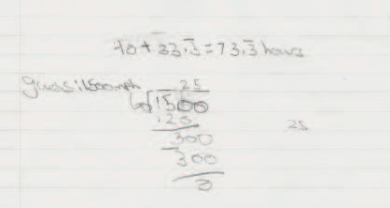
$$\frac{x}{50} + \frac{x}{60} = 36$$

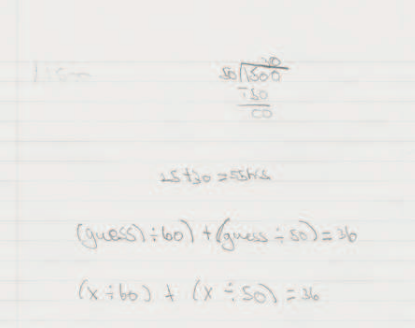
Example A: The Dreaded Word Problem

Here's some student work that shows how the process develops:



Example A: The Dreaded Word Problem





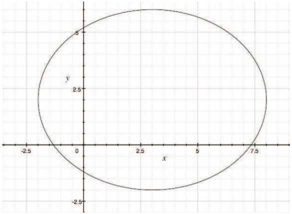
Example B: Equations for Lines

The phenomenon was first noticed in precalculus . . .
Graph

$$16x^2 - 96x + 25y^2 - 100y - 156 = 0$$

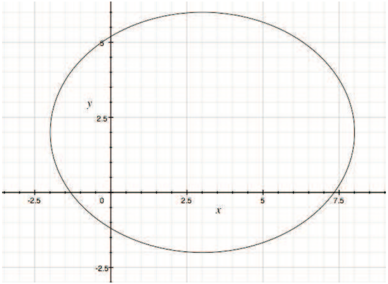
$$16x^2 - 96x + 25y^2 - 100y - 156 = 0 \Rightarrow \frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$$

$$\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1 \Rightarrow$$



Example B: Equations for Lines

$$\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$$



Is $(7.5, 3.75)$ on the graph?

This led to the idea that "equations are point testers."

Example B: Equations for Lines

- Suppose a student, new to algebra and with no formulas in tow, is asked to find the equation of the vertical line ℓ that passes through $(5, 4)$.
- Students can draw the line, and, just as in the word problem example, they can guess at some points and check to see if they are on ℓ.
- Trying some points like $(5, 1)$, $(3, 4)$, $(2, 2)$, and $(5, 17)$ leads to a generic guess-checker:

  To see if a point is on ℓ, you check that its x-coordinate is 5.

- This leads to a guess-checker: $x \stackrel{?}{=} 5$ and the equation

$$x = 5$$

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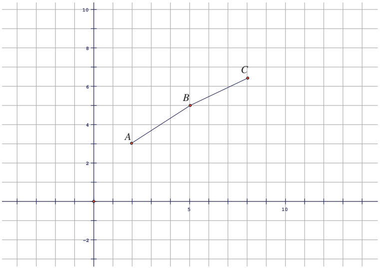
Example B: Equations for Lines

- What about lines for which there is no simple guess-checker? The idea is to find a geometric characterization of such a line and then to develop a guess-checker based on that characterization. One such characterization uses *slope*.
- In first-year algebra, students study slope, and one fact about slope that often comes up is that three points on the coordinate plane, not all on the same vertical line, are collinear if and only if the slope between any two of them is the same.

Example B: Equations for Lines

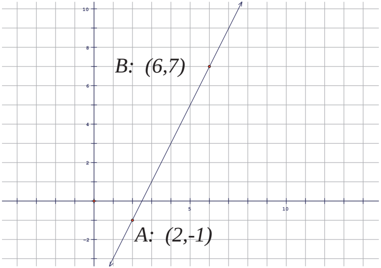
If we let $m(A, B)$ denote the slope between A and B (calculated as change in y-height divided by change in x-run), then the collinearity condition can be stated like this:

Basic assumption: A, B, and C are collinear $\Leftrightarrow m(A, B) = m(B, C)$



Example B: Equations for Lines

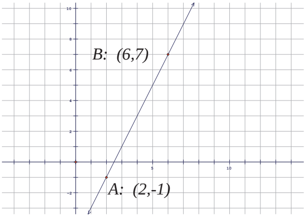
What is an equation for $\ell = \overleftrightarrow{AB}$ if $A = (2, -1)$ and $B = (6, 7)$?



Try some points, keeping track of the steps. . .

Example B: Equations for Lines

- $A = (2, -1)$ and $B = (6, 7)$
- $m(A, B) = 2$



- Test $C = (3, 4)$:
  $m(C, B) = \frac{4-7}{3-6} \stackrel{?}{=} 2 \Rightarrow$ Nope
- Test $D = (5, 5)$:
  $m(D, B) = \frac{5-7}{5-6} \stackrel{?}{=} 2 \Rightarrow$ Yup
- The "guess-checker?"
  Test $P = (x, y)$:
  $m(P, B) = \frac{y-7}{x-6} \stackrel{?}{=} 2$

And an equation is $\frac{y-7}{x-6} = 2$

Other Examples Where This Habit Is Useful

- Finding lines of best fit
- Building expressions ("*three less than a number*")
- Fitting functions to tables of data
- Deriving the quadratic formula
- Establishing identities in Pascal's triangle
- Using recursive definitions in a CAS or spreadsheet

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Monthly Payments on a Loan

Suppose you want to buy a car that costs \$10,000. You don't have much money, but you can put \$1000 down and pay \$230 per month. The interest rate is 5%, and the dealer wants the loan paid off in two years. Can you afford the car?

This leads to the question

"How does a bank figure out the monthly payment on a loan?"

or

"How does a bank figure out the balance you owe at the end of the month?"

Monthly Payments on a Loan

Take 1

What you owe at the end of the month is what you owed at the start of the month minus your monthly payment.

$$b(n, m) = \begin{cases} 9000 & \text{if } n = 0 \\ b(n-1, m) - m & \text{if } n > 0 \end{cases}$$

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Monthly Payments on a Loan

Take 2

> *What you owe at the end of the month is what you owed at the start of the month,* ***plus*** $\frac{1}{12}$ ***of the yearly interest on that amount****, minus your monthly payment.*

$$b(n,m) = \begin{cases} 9000 & \text{if } n = 0 \\ b(n-1,m) + \frac{.05}{12} b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

Students can then use successive approximation to find m so that

$$b(24,m) = 0$$

Monthly Payments on a Loan

Except . . .

$$b(4) = b(3) + \frac{.05}{12} \cdot b(3) - 250$$

$$b(2) + \frac{.05}{12} \cdot b(2) - 250 \qquad\qquad b(2) + \frac{.05}{12} \cdot b(2) - 250$$

It takes too much !$#& work.

Monthly Payments on a Loan

Take 3: Algebra to the rescue!

$$b(n,m) = \begin{cases} 9000 & \text{if } n = 0 \\ b(n-1,m) + \frac{.05}{12} b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

becomes

$$b(n,m) = \begin{cases} 9000 & \text{if } n = 0 \\ \left(1 + \frac{.05}{12}\right) b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

Students can *now* use successive approximation to find m so that

$$b(24,m) = 0$$

Monthly Payments on a Loan

Project: Pick an interest rate and keep it constant. Suppose you want to pay off a car in 24 months. Investigate how the monthly payment changes with the cost of the car:

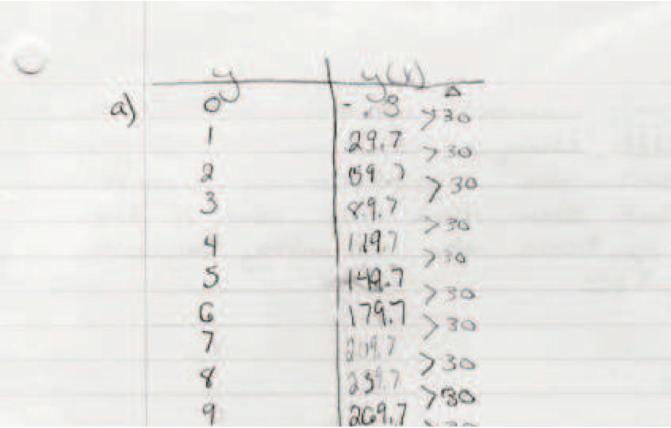
Cost of car (in thousands of dollars)	Monthly payment
10	
11	
12	
13	
14	
15	
⋮	⋮

Monthly Payments on a Loan

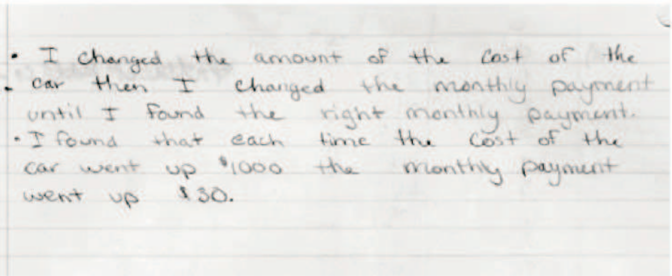
Cost of car (in thousands of dollars)	Monthly payment
10	
11	
12	
13	
14	
15	
⋮	⋮

Describe a pattern in the table. Use this pattern to find either a closed form or a recursive rule that lets you calculate the monthly payment in terms of the cost of the car in thousands of dollars. Model your function with your CAS and use the model to find the monthly payment on a $26000 car.

Monthly Payments on a Loan

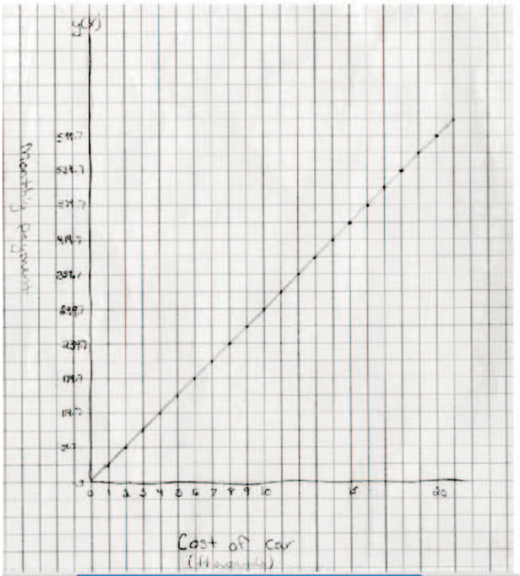


MONTHLY PAYMENTS ON A LOAN



• I changed the amount of the cost of the car then I changed the monthly payment until I found the right monthly payment.
• I found that each time the cost of the car went up $1000 the monthly payment went up $30.

Monthly Payments on a Loan



Monthly Payments on a Loan

Students can use a CAS to model the problem *generically*: the balance at the end of 24 months with a monthly payment of m can be found by entering

$$b(24, m)$$

in the calculator:

But *why* is it linear?

Monthly Payments on a Loan

But *why* is it linear?

Suppose you borrow \$9000 at 5% interest. Then you are experimenting with this function:

$$b(n,m) = \begin{cases} 9000 & \text{if } n = 0 \\ (1 + \frac{.05}{12}) \cdot b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

Notice that

$$1 + \frac{.05}{12} = \frac{12.05}{12}$$

Call this number q. So, the function now looks like:

$$b(n,m) = \begin{cases} 9000 & \text{if } n = 0 \\ q \cdot b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

where q is a constant (chunking, again).

Monthly Payments on a Loan

Then at the end of n months, you could unstack the calculation as follows:

$$
\begin{aligned}
b(n,m) &= q \cdot b(n-1,m) - m \\
&= q\,[q \cdot b(n-2,m) - m] - m \\
&\quad = q^2 \cdot b(n-2,m) - qm - m \\
&= q^2\,[q \cdot b(n-3,m) - m] - qm - m \\
&\quad = q^3 \cdot b(n-3,m) - q^2 m - qm - m \\
&\quad \vdots \\
&= q^n \cdot b(0,m) - q^{n-1}m - q^{n-2}m - \cdots - q^2 m - qm - m \\
&= 9000 \cdot q^n - m(q^{n-1} + q^{n-2} + \cdots + q^2 + q + 1)
\end{aligned}
$$

Monthly Payments on a Loan

Algebra 2 students know (very well) the "cyclotomic identity:"

$$(q-1)(q^{n-1}+q^{n-2}+\cdots+q^2+q+1)=q^n-1$$

or

$$q^{n-1}+q^{n-2}+\cdots+q^2+q+1=\frac{q^n-1}{q-1}$$

Applying it, you get

$$\begin{aligned} b(n,m) &= 9000\cdot q^n - m(q^{n-1}+q^{n-2}+\cdots+q^2+q+1) \\ &= 9000\,q^n - m\frac{q^n-1}{q-1} \end{aligned}$$

Setting $b(n,m)$ equal to 0 gives an explicit relationship between m and the cost of the car. . .

Monthly Payments on a Loan

$$m = 9000\,\frac{(q-1)q^n}{q^n - 1}$$

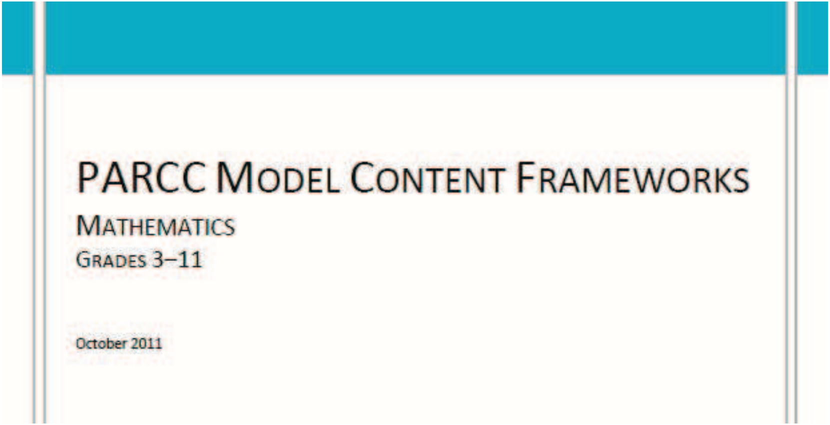
or, in general,

$$\text{monthly payment} = \text{cost of car} \times \frac{(q-1)q^n}{q^n - 1}$$

where n is the term of the loan and

$$q = 1 + \frac{\text{interest rate}}{12}$$

PARCC Model Content Frameworks



PARCC MODEL CONTENT FRAMEWORKS

MATHEMATICS

GRADES 3–11

October 2011

http://www.parcconline.org/parcc-content-frameworks

PARCC Model Content Frameworks: Elements

1. **General analysis of the high school standards:** analysis that bears on all courses and/or is independent of any particular organization of the standards into courses.

    - Examples of Opportunities for Connections among Standards, Clusters, Domains or Conceptual Categories.
    - Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices.
    - Examples of Content Standards that Apply to Two or More High School Courses.

PARCC Model Content Frameworks: Elements

2. **Course-specific analysis of the high school standards:** analysis presented with a view toward two possible high school course sequences.

   Each course is introduced with a high-level narrative. This narrative gives a sense of overall course goals. Then:

   - Examples of Key Advances from Previous Grades or Courses.
   - Fluency Recommendations.
   - Discussion of Mathematical Practices in Relation to Course Content.

PARCC: Practices in Algebra I

Two overarching practices relevant to Algebra I are:

- Make sense of problems and persevere in solving them.
- Model with mathematics.

Indeed, other mathematical practices in Algebra I might be seen as contributing specific elements of these two. The intent of the following set is not to decompose the above mathematical practices into component parts but rather to show how the mathematical practices work together.

PARCC: Practices in Algebra I

- **Reason abstractly and quantitatively.** This practice standard refers to one of the hallmarks of algebraic reasoning, the process of decontextualization and contextualization. Much of elementary algebra involves creating abstract algebraic models of problems and then transforming the models via algebraic calculations to reveal properties of the problems.
- **Use appropriate tools strategically.** Spreadsheets, a function modeling language, graphing tools and many other technologies can be used strategically to gain understanding of the ideas expressed by individual content standards and to model with mathematics.

PARCC: Practices in Algebra I

- **Attend to precision.** In algebra, the habit of using precise language is not only a mechanism for effective communication but also a tool for understanding and solving problems. Describing an idea precisely helps students understand the idea in new ways.
- **Look for and make use of structure.** For example, writing $49x^2 + 35x + 6$ as $(7x)^2 + 5(7x) + 6$, a practice many teachers refer to as “chunking,” highlights the structural similarity between this expression and $z^2 + 5z + 6$, leading to a factorization of the original: $((7x) + 3))((7x) + 2)$.

PARCC: Practices in Algebra I

- **Look for and express regularity in repeated reasoning.** Creating equations or functions to model situations is harder for many students than working with the resulting expressions. An effective way to help students develop the skill of describing general relationships is to work through several specific examples and then express what they are doing with algebraic symbolism. For example, when comparing two different text messaging plans, many students who can compute the cost for a given number of minutes have a hard time writing general formulas that express the cost of each plan for any number of minutes.

PARCC: Practices in Algebra I

- **Look for and express regularity in repeated reasoning, cont'd.** . . . Constructing these formulas can be facilitated by methodically calculating the cost for several different input values and then expressing the steps in the calculation, first in words and then in algebraic symbols. Once such expressions are obtained, students can find the break-even point for the two plans, graph the total cost against the number of messages sent and make a complete analysis of the two plans.

PARCC: Practices in Geometry

- **Reason abstractly and quantitatively.** Abstraction is used in geometry when, for example, students use a diagram of a specific isosceles triangle as an aid to reason about all isosceles triangles. Quantitative reasoning in geometry involves the real numbers in an essential way: Irrational numbers show up in work with the Pythagorean theorem, area formulas often depend (subtly and informally) on passing to the limit and real numbers are an essential part of the definition of dilation. The proper use of units can help students understand the effect of dilation on area and perimeter.

PARCC: Practices in Geometry

- **Construct viable arguments and critique the reasoning of others.** While all of high school mathematics should work to help students see the importance and usefulness of deductive arguments, geometry is an ideal arena for developing the skill of creating and presenting proofs. One reason is that conjectures about geometric phenomena are often about infinitely many cases at once—for example, *every* angle inscribed in a semicircle is a right angle—so such results cannot be established by checking every case.
- **Use appropriate tools strategically.** Dynamic geometry environments can help students look for invariants in a whole class of geometric constructions, and the constructions in such environments can sometimes lead to an idea behind a proof of a conjecture.

PARCC: Practices in Geometry

- **Attend to precision.** Teachers might use the activity of creating definitions as a way to help students see the value of precision. While this is possible in every course, the activity has a particularly visual appeal in geometry. For example, a class can build the definition of quadrilateral by starting with a rough idea ("four sides"), gradually refining the idea so that it rules out figures that do not fit the intuitive idea. Another place in geometry where precision is necessary and useful is in the refinement of conjectures so that initial conjectures that are not correct can be salvaged—two angle measures and a side length do not determine a triangle, but a certain configuration of these parts leads to the angle-side-angle theorem.

PARCC: Practices in Geometry

- **Look for and make use of structure.** Seeing structure in geometric configurations can lead to insights and proofs. This often involves the creation of auxiliary lines not originally part of a given figure. Two classic examples are the construction of a line through a vertex of a triangle parallel to the opposite side as a way to see that the angle measures of a triangle add to 180° and the introduction of a symmetry line in an isosceles triangle to see that the base angles are congruent. Another kind of hidden structure makes use of area as a device to establish results about proportions, such as the important theorem (and its converse) that a line parallel to one side of a triangle divides the other two sides proportionally.

PARCC: Practices in Algebra II

- **Construct viable arguments and critique the reasoning of others.** As in geometry, there are central questions in advanced algebra that cannot be answered definitively by checking evidence. There are important results about all functions of a certain type—the factor theorem for polynomial functions, for example—and these require general arguments. Deciding whether two functions are equal on an infinite set cannot be settled by looking at tables or graphs; it requires arguments of a different sort.

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PARCC: Practices in Algebra II

- **Attend to precision.** As in the previous two courses, the habit of using precise language is not only a tool for effective communication but also a means for coming to understanding. For example, when investigating loan payments, if students can articulate something like, "What you owe at the end of a month is what you owed at the start of the month, plus $\frac{1}{12}$th of the yearly interest on that amount, minus the monthly payment," they are well along a path that will let them construct a recursively defined function for calculating loan payments.

PARCC: Practices in Algebra II

- **Look for and make use of structure.** The structure theme in Algebra I centered on seeing and using the structure of algebraic expressions. This continues in Algebra II, where students delve deeper into transforming expressions in ways that reveal meaning.

  The example given in the standards—that $x^4 - y^4$ can be seen as the difference of squares—is typical of this practice. This habit of seeing subexpressions as single entities will serve students well in areas such as trigonometry, where, for example, the factorization of $x^4 - y^4 \ldots$ can be used to show that the functions $x \mapsto \cos^4 x - \sin^4 x$ and $x \mapsto \cos^2 x - \sin^2 x$ are, in fact, equal.

PARCC: Practices in Algebra II

- **Structure, cont'd.** In addition, the standards call for attention to the structural similarities between polynomials and integers. The study of these similarities can be deepened in Algebra II: Like integers, polynomials have a division algorithm, and division of polynomials can be used to understand the factor theorem, transform rational expressions, help solve equations, and factor polynomials.

PARCC: Practices in Algebra II

- **Look for and express regularity in repeated reasoning.** Algebra II is where students can do a more complete analysis of sequences, especially arithmetic and geometric sequences, and their associated series. Developing recursive formulas for sequences is facilitated by the practice of abstracting regularity for how you get from one term to the next and then giving a precise description of this process in algebraic symbols. . . .

PARCC: Practices in Algebra II

- **Repeated reasoning, cont'd.** The same thinking—finding and articulating the rhythm in calculations—can help students analyze mortgage payments, and the ability to get a closed form for a geometric series lets them make a complete analysis of this topic. This practice is also a tool for using difference tables to find simple functions that agree with a set of data.

PARCC: Practices in Algebra II

- **Look for and express regularity in repeated reasoning, cont'd.** Algebra II is a course in which students can learn some technical methods for performing algebraic calculations and transformations, but sense-making is still paramount. For example, analyzing Heron's formula from geometry lets one connect the zeros of the expression to the degenerate triangles. As in Algebra I, the modeling practice is ubiquitous in Algebra II, enhanced by the inclusion of exponential and logarithmic functions as modeling tools. . . .

$$A = \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

IN SUMMARY

Whew

Some Conclusions

- The Standards for Mathematical Practice elevate the methods used by mathematicians to the same level of importance as the results of those methods.
- Developing these practices to the point where they are invoked as habits takes sustained and concentrated effort—from teachers and students—across P–12.
- And *applying* these practices in mathematical contexts takes sustained and concentrated effort—from teachers and students—across P–12. We need more examples of this.
- The eight standards are foundations—not capstones—for the practice of mathematics.

Resources

- The Institute for Mathematics and Education
  - http://ime.math.arizona.edu/commoncore/
- Tools for the Common Core
  - http://commoncoretools.me/
- PARCC Model Content Frameworks
  - http://www.parcconline.org/
- Focus in High School Mathematics: Reasoning and Sense Making in Algebra
  - http://www.nctm.org/
- Patterns in Practice Blog
  - http://patternsinpractice.wordpress.com/

Some Conclusions

Time for Questions and Comments.

Thanks

Al Cuoco

Center for Mathematics Education

Education Development Center, Inc.

acuoco@edc.org

http://edc.org/cme/mpi