$$\frac{1}{9801} = 0.000102030405060708091011112\ldots \qquad 1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{21} + \frac{1}{420}$$

$\frac{1}{n}$

NEAT STUFF ABOUT UNIT FRACTIONS

Al Cuoco
Center for Mathematics Education, EDC
(with help from the Progressions Project at IM&E)

EDC Learning transforms lives.

Outline

Are there any rectangles

whose area and perimeter have the same numerical value?

Can you write $\frac{1}{2}$

as the sum of two "unit fractions"?

$$\frac{1}{2} = \frac{1}{a} + \frac{1}{b}$$

A basic engine for progress in mathematics

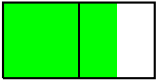
- Ask a second grader "What is $\frac{1}{2}$?"
- Ask a high school sophomore "What is cosine?"
- Ask a calculus student "What is the derivative?"
- The methods of one generation become the objects of study for the next.

A BASIC ENGINE FOR PROGRESS IN MATHEMATICS

- Ask a second grader "What is $\frac{1}{2}$?"
- As a high school sophomore "What is cosine?"
- As a calculus student "What is the derivative?"
- The methods of one ~~grade level~~ generation become the objects of study for the next.
- And the path from method to object is slow, non-sequential, and two-way.

3.NF.1. Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

The importance of specifying the whole



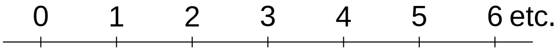
Without specifying the whole it is not reasonable to ask what fraction is represented by the shaded area. If the left square is the whole, the shaded area represents the fraction $\frac{3}{2}$*; if the entire rectangle is the whole, the shaded area represents* $\frac{3}{4}$.

3.NF.2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.

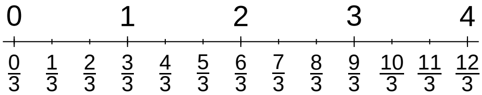
A. Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.

B. Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

The number line



The number line marked off in thirds



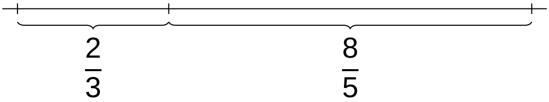
4.NF.3. Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

A. Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.

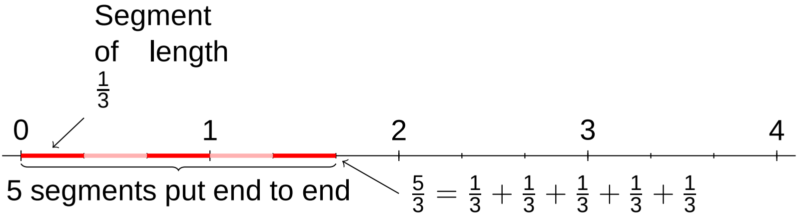
B. Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.

C. Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.

Representation of $\frac{2}{3} + \frac{8}{5}$ as a length



Using the number line to see that $\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$



$$\begin{aligned}
\frac{7}{5}+\frac{4}{5} &= \overbrace{\frac{1}{5}+\cdots\frac{1}{5}}^{7}+\overbrace{\frac{1}{5}+\cdots\frac{1}{5}}^{4} \\
&= \frac{\overbrace{1+1+\cdots+1}^{7+4}}{5} \\
&= \frac{7+4}{5}.
\end{aligned}$$

THE SAME IDEA IS USED LATER FOR EXPONEITATION

$$\left(\frac{1}{5}\right)^7\left(\frac{1}{5}\right)^4 = \overbrace{\frac{1}{5}\times\cdots\times\frac{1}{5}}^{7}\times\overbrace{\frac{1}{5}\times\cdots\times\frac{1}{5}}^{4}$$

$$= \overbrace{\frac{1}{5}\times\cdots\times\frac{1}{5}}^{7+4}$$

$$= \left(\frac{1}{5}\right)^{(7+4)}.$$

EDC Learning transforms lives.

4.NF.4. Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

A Understand a fraction a/b as a multiple of $1/b$.

B Understand a multiple of a/b as a multiple of $1/b$, and use this understanding to multiply a fraction by a whole number.

$$\frac{7}{5} = 7 \times \frac{1}{5}, \qquad \frac{11}{3} = 11 \times \frac{1}{3}.$$

$$3 \times \frac{2}{5} \quad \text{as} \quad 3 \times \left(2 \times \frac{1}{5}\right) = (3 \times 2) \times \frac{1}{5} = 6 \times \frac{1}{5} = \frac{6}{5}$$

$$3 \times \frac{2}{5} \quad \text{as} \quad \frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{3 \times 2}{5} = \frac{6}{5}.$$

4.NF.5. Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.

4.NF.6. Use decimal notation for fractions with denominators 10 or 100.

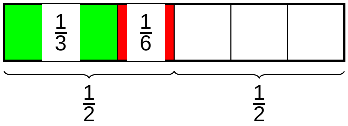
$$\frac{3}{10} = \frac{3 \times 10}{10 \times 10} = \frac{30}{100}.$$

$$\frac{3}{10} + \frac{27}{100} = \frac{30}{100} + \frac{27}{100} = \frac{57}{100}.$$

5.NF.1. Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.

5.NF.2. Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers.

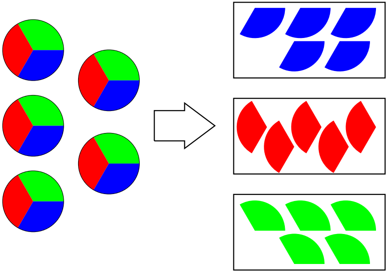
Using a fraction strip to show that $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$



5.NF.3. Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

How to share 5 objects equally among 3 shares:

$$5 \div 3 = 5 \times \frac{1}{3} = \frac{5}{3}$$

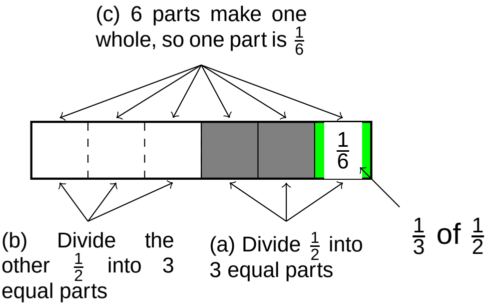


If you divide 5 *objects equally among* 3 *shares, each of the* 5 *objects should contribute* $\frac{1}{3}$ *of itself to each share. Thus each share consists of* 5 *pieces, each of which is* $\frac{1}{3}$ *of an object, and so each share is* $5 \times \frac{1}{3} = \frac{5}{3}$ *of an object.*

5.NF.4. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

A Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$.

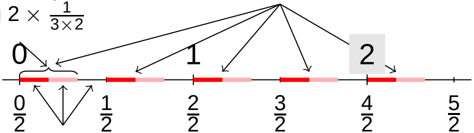
Using a fraction strip to show that $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$



Using a number line to show that $\frac{2}{3} \times \frac{5}{2} = \frac{2\times 5}{3\times 2}$

(b) Form a segment from 2 parts, making $2 \times \frac{1}{3\times 2}$

(c) There are 5 of the $\frac{1}{2}$s, so the segments together make $5 \times (2 \times \frac{1}{3\times 2}) = \frac{2\times 5}{3\times 2}$



(a) Divide each $\frac{1}{2}$ into 3 equal parts, so each part is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{3\times 2}$

A standard football field is 100 yards long from end zone to end zone. Vince measures a field and finds that it is five feet shorter than a standard field. How long is the shorter field?

1.13 Adding and Subtracting Fractions

In this lesson, you will learn to add and subtract fractions with like and unlike denominators.

Minds in Action episode 2

Tony and Sasha work on Exercise 13 from Lesson 1.12.

A standard football field is 100 yards long from end zone to end zone. Vince measures a field and finds that it is five feet shorter than a standard field. How long is the shorter field?

Tony I've got the answer. It's 295 feet.

Sasha Two hundred ninety-five? I got 95. That seems fine to me.

Tony No, the answer is definitely 295 feet.

Sasha How can the field be 295 feet long? The regular field is 100 yards. Oh, I see. You got 295 feet. Now I see what I did wrong.

To find the length of the shorter field, you can write this subtraction. When Sasha subtracts the numbers, the calculation does not make sense. The units are not the same!

$$\begin{array}{r} 100 \text{ yards} \\ -\ \ 5 \text{ feet} \\ \hline \blacksquare \end{array}$$

There are two ways to change the units. You can convert 100 yards to feet, or convert 5 feet to yards. Converting yards to feet gives you this subtraction problem.

	300 feet
−	5 feet
	■ feet

There are three feet in a yard, so 100 yards equal 300 feet. You can also convert 5 feet to $\frac{5}{3}$ yards. Then subtract that amount from 100 yards. The result is in yards, instead of feet.

You can use this method to add and subtract fractions. For example, you can write the sum $\frac{2}{3} + \frac{1}{5}$ like this. You need to convert the denominators of both fractions to a common unit.

	2 thirds
+	1 fifth
	■

Here are some ways to write $\frac{2}{3}$ as an equivalent fraction.

$$\frac{2}{3} = \frac{4}{6} \qquad \frac{4}{6} = \frac{6}{9} \qquad \frac{6}{9} = \frac{8}{12} \qquad \frac{8}{12} = \frac{10}{15} \qquad \frac{10}{15} = \frac{12}{18}$$

Here are some equivalent fractions for $\frac{1}{5}$.

$$\frac{1}{5} = \frac{2}{10} \qquad \frac{2}{10} = \frac{3}{15} \qquad \frac{3}{15} = \frac{4}{20} \qquad \frac{4}{20} = \frac{5}{25}$$

Fifteenths is the lowest possible match, or least common denominator. What other matches are possible?

In the list below there is a matching unit, fifteenths. Two thirds is equivalent to ten fifteenths, and one fifth is equivalent to three fifteenths. Now that you have a common unit, you can find the sum.

	10 fifteenths
+	3 fifteenths
	13 fifteenths

$$\frac{2}{3} + \frac{1}{5} = \frac{10}{15} + \frac{3}{15} = \frac{13}{15}$$

Unit chunks

A. $2\left(\frac{1}{5}\right) - \left(\frac{1}{5}\right)$

B. $2\left(\frac{1}{5}\right) - \left(\frac{1}{5}\right) + 4\left(\frac{1}{5}\right) - 3\left(\frac{1}{5}\right)$

C. $2\left(\frac{1}{5}\right) - \left(\frac{1}{5}\right) + 4\left(\frac{1}{5}\right) - 3\left(\frac{1}{5}\right) + 6\left(\frac{1}{5}\right) - 5\left(\frac{1}{5}\right)$

D. $2\left(\frac{1}{5}\right) - \left(\frac{1}{5}\right) + 4\left(\frac{1}{5}\right) - 3\left(\frac{1}{5}\right) + 6\left(\frac{1}{5}\right) - 5\left(\frac{1}{5}\right) + 8\left(\frac{1}{5}\right) - 7\left(\frac{1}{5}\right)$

E. $2\left(\frac{1}{5}\right) - \left(\frac{1}{5}\right) + 4\left(\frac{1}{5}\right) - 3\left(\frac{1}{5}\right) + 6\left(\frac{1}{5}\right) - 5\left(\frac{1}{5}\right) + 8\left(\frac{1}{5}\right) - 7\left(\frac{1}{5}\right) + 10\left(\frac{1}{5}\right) - 9\left(\frac{1}{5}\right)$

"Unit" expressions

5. Simplify.

a. $2(x + 2) - (x + 2)$

b. $2(x + 2) - (x + 2) + 4(x + 2) - 3(x + 2)$

c. $2(x + 2) - (x + 2) + 4(x + 2) - 3(x + 2) + 6(x + 2) - 5(x + 2)$

d. $2(x + 2) - (x + 2) + 4(x + 2) - 3(x + 2) + 6(x + 2) - 5(x + 2) + 8(x + 2) - 7(x + 2)$

e. $2(x + 2) - (x + 2) + 4(x + 2) - 3(x + 2) + 6(x + 2) - 5(x + 2) + 8(x + 2) - 7(x + 2) + 10(x + 2) - 9(x + 2)$

f. Evaluate each simplified expression for $x = -2$. What is the pattern in your results? Explain.

Are there any rectangular boxes

whose surface area and volume have the same numerical value?

"UNIT" BASES

$$
\begin{aligned}
325 &= 7 \cdot 46 + \mathbf{3} \\
&\swarrow \\
46 &= 7 \cdot 6 + \mathbf{4} \\
&\swarrow \\
6 &= 7 \cdot 0 + \mathbf{6}
\end{aligned}
$$

So

$$
\begin{aligned}
325 &= 3 + 7(46) \\
&= 3 + 7\,(4 + 7 \cdot 6) \\
&= 3 + 7 \cdot 4 + 7^2 \cdot 6
\end{aligned}
$$

"UNIT" POLYNOMIALS

Write

$$f(x) = x^4 - 5x^3 + 3x - 1$$

in terms of the "unit" $x - 3$

"Unit" polynomials

$\text{polyRemainder}\left(x^4-5\cdot x^3+3\cdot x-1,x-3\right)$	$^-46$
$\text{polyQuotient}\left(x^4-5\cdot x^3+3\cdot x-1,x-3\right)$	$x^3-2\cdot x^2-6\cdot x-15$
$\text{polyRemainder}\left(x^3-2\cdot x^2-6\cdot x-15,x-3\right)$	$^-24$
$\text{polyQuotient}\left(x^3-2\cdot x^2-6\cdot x-15,x-3\right)$	x^2+x-3
$\text{polyRemainder}\left(x^2+x-3,x-3\right)$	9
$\text{polyQuotient}\left(x^2+x-3,x-3\right)$	$x+4$
$\text{polyRemainder}\left(x+4,x-3\right)$	7
$\text{polyQuotient}\left(x+4,x-3\right)$	1

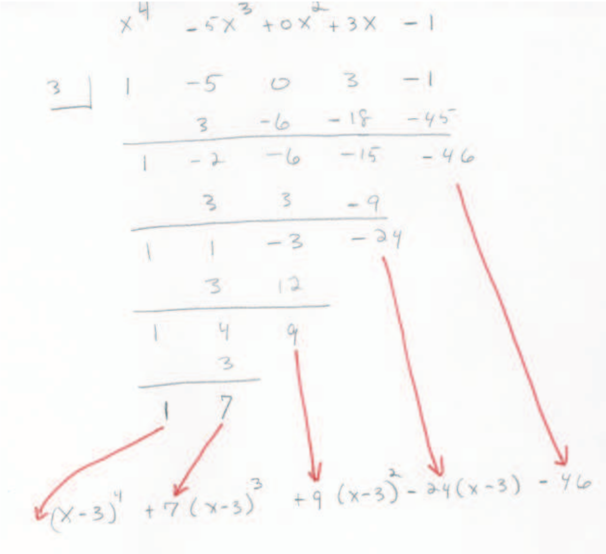
“UNIT” POLYNOMIALS

Now put it all together:

$$
\begin{aligned}
f(x) &= -46 + (x-3)(\underbrace{x^3 - 2x^2 - 6x - 15}_{q(x)}) \\
&= -46 + (x-3)\left(-24 + (x-3)(x^2 + x - 3\right) \\
&= -46 - 24(x-3) + (x-3)^2\left(\underbrace{x^2 + x - 3}_{q_1(x)}\right) \\
&= -46 - 24(x-3) + (x-3)^2\left(9 + (x-3)(x+4)\right) \\
&= -46 - 24(x-3) + 9(x-3)^2 + (x-3)^3\left(\underbrace{x+4}_{q_2(x)}\right) \\
&= -46 - 24(x-3) + 9(x-3)^2 + (x-3)^3\left(7 + (x-3)\right) \\
&= -46 - 24(x-3) + 9(x-3)^2 + 7(x-3)^3 + (x-3)^4
\end{aligned}
$$

“Unit” polynomials

For those of us who enjoy a good calculation



"UNIT" POLYNOMIALS

Indeed:

$$\text{taylor}\left(x^4-5\cdot x^3+3\cdot x-1,x,4,3\right) \qquad -46-24\cdot(x-3)+9\cdot(x-3)^2+7\cdot(x-3)^3+(x-3)^4$$

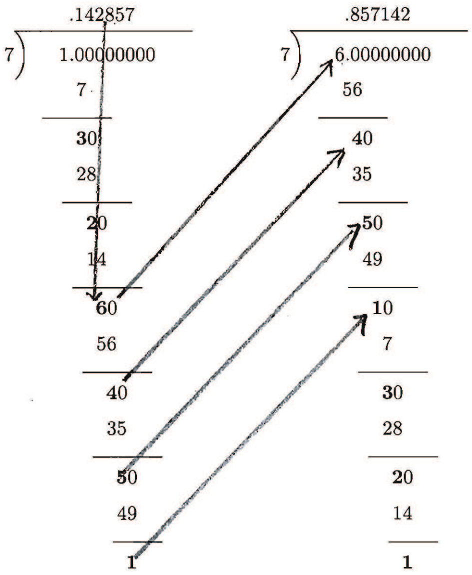
Let's figure out the decimal expansions for

- $\frac{1}{7}$
- $\frac{2}{7}$
- $\frac{3}{7}$
- $\frac{4}{7}$
- $\frac{5}{7}$
- $\frac{6}{7}$

Let's figure out the decimal expansions for

- $\frac{1}{7} = .142857142857\ldots$
- $\frac{2}{7} = .285714285714\ldots$
- $\frac{3}{7} = .428571428571\ldots$
- $\frac{4}{7} = .571428571428\ldots$
- $\frac{5}{7} = .714285714285\ldots$
- $\frac{6}{7} = .857142857142\ldots$

```
        .142857
     ___________
  7 ) 1.00000000
      7
      ---
       30
       28
       ---
        20
        14
        ---
         60
         56
         ---
          40
          35
          ---
           50
           49
           ---
            1
```


.142857
7) 1.00000000
7
30
28
20
14
60
56
40
35
50
49
1
.857142
7) 6.00000000
56
40
35
50
49
10
7
30
28
20
14
1

FOOD FOR THOUGHT

- Are there any unit fractions whose decimal expansions terminate?
- Are there any unit fractions whose decimal expansions don't repeat?
- What's the longest possible period for the decimal expansion of $\frac{1}{n}$?
- What are the *possible* periods for the decimal expansion of $\frac{1}{n}$?

$$
\begin{array}{r}
.076923\ldots \\
13\overline{)\,1.000000000} \\
\underline{0} \\
\mathbf{10}0 \\
\underline{91} \\
\mathbf{9}0 \\
\underline{78} \\
\mathbf{12}0 \\
\underline{117} \\
\mathbf{3}0 \\
\underline{26} \\
\mathbf{4}0 \\
\underline{39} \\
\mathbf{1}
\end{array}
$$

We have a theorem (sort of)

The period for the decimal expansion for $\frac{1}{n}$ is the smallest power of 10 that leaves a remainder of 1 when divided by n.

If n is not divisible by 2 or 5, the period for the decimal expansion for $\frac{1}{n}$ is the smallest power of 10 that leaves a remainder of 1 when divided by n.

Erdős–Straus (1948)

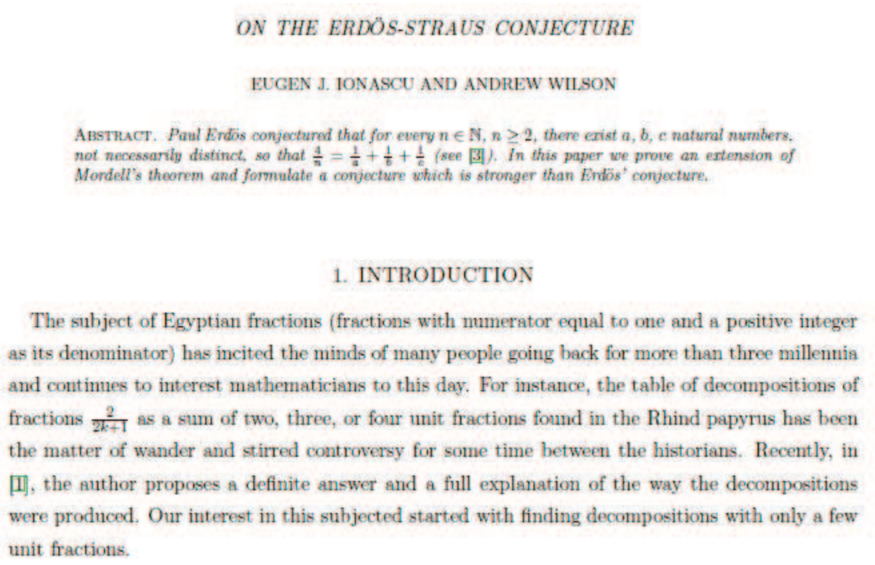
Every fraction of the form $\frac{4}{n}$ $(n \geq 4)$

is the sum of three unit fractions

$$\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{10} \qquad \frac{4}{11} = \frac{1}{3} + \frac{1}{44} + \frac{1}{132}$$

(verified for all $n < 10^{14}$)

From a paper (2010)



ON THE ERDÖS-STRAUS CONJECTURE

EUGEN J. IONASCU AND ANDREW WILSON

Abstract. *Paul Erdős conjectured that for every $n \in \mathbb{N}$, $n \geq 2$, there exist a, b, c natural numbers, not necessarily distinct, so that $\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ (see [3]). In this paper we prove an extension of Mordell's theorem and formulate a conjecture which is stronger than Erdős' conjecture.*

1. INTRODUCTION

The subject of Egyptian fractions (fractions with numerator equal to one and a positive integer as its denominator) has incited the minds of many people going back for more than three millennia and continues to interest mathematicians to this day. For instance, the table of decompositions of fractions $\frac{2}{2k+1}$ as a sum of two, three, or four unit fractions found in the Rhind papyrus has been the matter of wander and stirred controversy for some time between the historians. Recently, in [1], the author proposes a definite answer and a full explanation of the way the decompositions were produced. Our interest in this subjected started with finding decompositions with only a few unit fractions.

Gauss (1801)

There are infinitely many primes p for which

the decimal expansion of $\frac{1}{p}$ has length $p-1$

I. TABLE

de fractions, dont les diviseurs sont des nombres premiers, réduites en décimales périodiques.

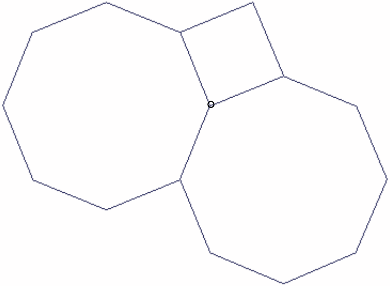
$1 : D =$	$0 + (10^{\varepsilon} - 1) : D \times 10^{\varepsilon} + (10^{\varepsilon} - 1) : D \times 10^{2\varepsilon} +$ &c. donc	ε ou $(D - 1) : \delta$
1 : 3 =	0, 3 &c.	1 = (3 — 1) : 2
1 : 7 =	0, 142857	6 = (7 — 1) : 1
1 : 11 =	0, 09	2 = (11 — 1) : 5
1 : 13 =	0, 076923	6 = (13 — 1) : 2
1 : 17 =	0, 0588235294117647	16 = (17 — 1) : 1
1 : 19 =	0, 052631578947368421	18 = (19 — 1) : 1
1 : 23 =	0, 0434782608695652173913	22 = (23 — 1) : 1
1 : 29 =	0, 0344827586206896551724137931	28 = (29 — 1) : 1
1 : 31 =	0, 032258064516129	15 = (31 — 1) : 2
1 : 37 =	0, 027	3 = (37 — 1) : 12
1 : 41 =	0, 02439	5 = (41 — 1) : 8
1 : 43 =	0, 023255813953488372093	21 = (43 — 1) : 2
1 : 47 =	0, 0212765957446808510638297872340425531914893617	46 = (47 — 1) : 1
1 : 53 =	0, 0188679245283	13 = (53 — 1) : 4
1 : 59 =	0, 0169491525423728813559322033898305084745762711864406779661	58 = (59 — 1) : 1
1 : 61 =	0, 016393442622950819672131147540983606557377049180327868852459.	60 = (61 — 1) : 1
1 : 67 =	0, 014925373134328358208955223880597	33 = (67 — 1) : 2

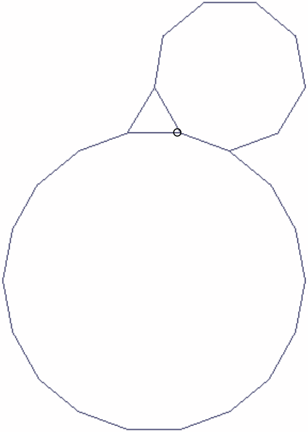
Gauss (in 1801):

“[This] is reduced for the most part to trial and error.”

Problems for the flight home

Which sets of three regular polygons can “tile a corner?”





Problems for the flight home

- What is the decimal expansion for $\frac{1}{9899}$?
- What's up with this?

$$\frac{1}{9801} = 0.0001020304050607080910111 2\ldots$$

Learning transforms lives.

Resources

- The Institute for Mathematics and Education
  - http://ime.math.arizona.edu/commoncore/
- Tools for the Common Core
  - http://commoncoretools.me/
- PARCC Model Content Frameworks
  - http://www.parcconline.org/parcc-content frameworks
- Patterns in Practice Blog
  - http://patternsinpractice.wordpress.com/
- Smarter Balance
  - http://www.k12.wa.us/smarter/

THANKS

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acuoco @ edc.org

Slides posted on

http://cmeproject.edc.org/