CCSS and High School Mathematics Curricula

The case of the *CME Project*

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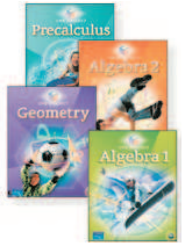
Mathematics and the Common Core, February, 2011

`www.edc.org/cmeproject`

Outline

THE CME PROJECT: BRIEF OVERVIEW

- An NSF-funded coherent 4-year curriculum
- Published by Pearson
- Follows the traditional American course structure
- Uses TI-Nspire technology to support mathematical thinking
- Organized around mathematical habits of mind



The Habits of Mind Approach

What mathematicians most wanted and needed from me was to learn my ways of thinking, and not in fact to learn my proof of the geometrization conjecture for Haken manifolds.

—William Thurston
On Proof and Progress in Mathematics

THE FUNDAMENTAL ORGANIZING PRINCIPLE OF *The CME Project*

The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more importantly, from developing the ways of thinking—the habits of mind—used to create the results.

—*The CME Project* Implementation Guide, p. 7 (2008)

Connecting Content to the Practice of Mathematics

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise. . . . Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

—CCSS, p. 8 (2010)

Common Core: Mathematical Practices

Eight attributes of mathematical proficiency:

1. **Make sense of complex problems and persevere in solving them.**
   Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals . . . They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution.
2. **Reason abstractly and quantitatively.**
   Mathematically proficient students make sense of the quantities and their relationships in problem situations. . . . Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize. . . and the ability to contextualize. . . .

Common Core: Mathematical Practices

3. **Construct viable arguments and critique the reasoning of others.**
   Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. . . .
4. **Model with mathematics.**
   Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later.

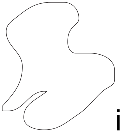
Common Core: Mathematical Practices

5. **Use appropriate tools strategically.**
   Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, ruler, protractor, calculator, spreadsheet, computer algebra system, statistical package, or dynamic geometry software. . . .
6. **Attend to precision.**
   Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. . . .

Common Core: Mathematical Practices

7. **Look for and make use of structure.**
   Mathematically proficient students look closely to discern a pattern or structure. . . . For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.
8. **Look for and express regularity in repeated reasoning.**
   By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1, 2)$ with slope 3, middle school students might abstract the equation $(y - 1)/(x - 2) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series.

Habits of Mind: Examples from *CME*

- Is there a line that cuts the area of  in half?
- Was there a time in your life when your height in inches was equal to your weight in pounds? (*Tom Banchoff*)
- Is the average of two averages the average of the lot?

Additional Core Principles

- **Textured emphasis.** We are careful to separate convention and vocabulary from matters of mathematical substance.
- **General purpose tools.** The methods and habits that students develop in high school should serve them well in their later work in mathematics and in their post-secondary endeavors.
- **Experience before formality.** Worked-out examples and careful definitions are important, but students need to grapple with ideas and problems *before* these things are brought to closure.

Additional Core Principles, continued

- **High expectations.** The vast majority of students have the capacity to think in ways that are characteristically mathematical.
- **A mathematical community.** Our writers, field testers, reviewers, and advisors come from all parts of the mathematics community: teachers, mathematicians, education researchers, technology developers, and administrators.
- **Connect school mathematics to the discipline.** Every chapter, lesson, problem, and example is written with an eye towards how it fits into the landscape of mathematics as a scientific discipline.

From a popular text ($\sim$ 1980)

“Factoring Pattern for $x^2 + bx + c$, c Negative”

Factor. Check by multiplying factors. If the polynomial is not factorable, write “prime.”

1. $a^2 + 4a - 5$	**2.** $x^2 - 2x - 3$	**3.** $y^2 - 5y - 6$
4. $b^2 + 2b - 15$	**5.** $c^2 - 11c - 10$	**6.** $r^2 - 16r - 28$
7. $x^2 - 6x - 18$	**8.** $y^2 - 10c - 24$	**9.** $a^2 + 2a - 35$
10. $k^2 - 2k - 20$	**11.** $z^2 + 5z - 36$	**12.** $r^2 - 3r - 40$
13. $p^2 - 4p - 21$	**14.** $a^2 + 3a - 54$	**15.** $y^2 - 5y - 30$
16. $z^2 - z - 72$	**17.** $a^2 - ab - 30b^2$	**18.** $k^2 - 11kd - 60d^2$
19. $p^2 - 5pq - 50q^2$	**20.** $a^2 - 4ab - 77b^2$	**21.** $y^2 - 2yz - 3z^2$
22. $s^2 + 14st - 72t^2$	**23.** $x^2 - 9xy - 22y^2$	**24.** $p^2 - pq - 72q^2$

From a Published Text (2010)

To factor a trinomial of the form $ax^2 + bx + c$ where $a > 0$, follow these steps:

$$\begin{array}{c|c|c} A & B & C \\ \hline F & H & D \\ \hline G & I & E \end{array}$$

1. Identify the values of a, b, c. Put a in Box A and c in Box B. Put the product of a and c in Box C.
2. List the factors of the number from Box C and identify the pair whose sum is b. Put the two factors you find in Box D and E.
3. Find the greatest common factor of Boxes A [sic] and E and put it in box G.

FROM A PUBLISHED TEXT (2010)

A	B	C
F	H	D
G	I	E

4. In Box F, place the number you multiply by Box G to get Box A.
5. In Box H, place the number you multiply by Box F to get Box D.
6. In Box I, place the number you multiply by Box G to get Box E.

Solution: The binomial factors whose product gives the trinomial are $(Fx + I)(Gx + H)$.

Factoring in *CME*

Factoring monic quadratics:

"Sum-Product" problems

$$x^2 + 14x + 48$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

so...

Find two numbers whose sum is 14 and whose product is 48.

$$(x + 6)(x + 8)$$

Factoring in *CME*

What about this one?

$$49x^2 + 35x + 6$$

$$\begin{aligned} 49x^2 + 35x + 6 &= (7x)^2 + 5(7x) + 6 \\ &= \clubsuit^2 + 5\clubsuit + 6 \\ &= (\clubsuit + 3)(\clubsuit + 2) \\ &= (7x + 3)(7x + 2) \end{aligned}$$

FACTORING IN *CME*

What about this one?

$$6x^2 + 31x + 35$$

$$\begin{aligned}
6(6x^2 + 31x + 35) &= (6x)^2 + 31(6x) + 210 \\
&= \clubsuit^2 + 31\clubsuit + 210 \\
&= (\clubsuit + 21)(\clubsuit + 10) \\
&= (6x + 21)(6x + 10) \\
&= 3(2x + 7) \cdot 2(3x + 5) \\
&= 6(2x + 7)(3x + 5) \quad \textit{so} \ldots
\end{aligned}$$

$$6(6x^2 + 31x + 35) = 6(2x + 7)(3x + 5)$$

FACTORING IN *CME*

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$$\cancel{6}(6x^2 + 31x + 35) = \cancel{6}(2x + 7)(3x + 5)$$

Factoring in *CME*

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$$6x^2 + 31x + 35 \; = \; (2x + 7)(3x + 5)$$

FACTORING IN *CME*

- This technique is perfectly general and can be used to transform a polynomial of any degree into one whose leading coefficient is 1.
- And it fits into the larger landscape of the *theory of equations* that shows how to use similar transformations to
  - remove terms
  - transform roots
  - derive “formulas” for equations of degree 3 and 4
  - extend the notion of *discriminant* to higher degrees

Other Examples Where Chunking Is Useful

- Completing the square and removing terms
- Solving trig equations
- Analyzing conics and other curves
- All over calculus
- Interpreting results from a computer algebra system

$\vdots$

Some Thorny Topics in Algebra

1. Students have trouble expressing generality with algebraic notation.
2. This is especially prevalent when they have to set up equations to solve word problems.
3. Many students have difficulty with slope, graphing lines, and finding equations of lines.
4. Building and using algebraic functions is another place where students struggle.

This list looks like a collection of disparate topics—using notation, solving word problems, ...

Some Thorny Topics in Algebra

But if one looks underneath the topics to the mathematical habits that would help students master them, one finds a remarkable similarity:

A key ingredient in such a mastery is the reasoning habit of seeking and expressing regularity in repeated calculations.

- This habit manifests itself when one is performing the same calculation over and over and begins to notice the "rhythm" in the operations.
- Articulating this regularity leads to a generic algorithm, typically expressed with algebraic symbolism, that can be applied to any instance and that can be transformed to reveal additional meaning, often leading to a solution of the problem at hand.

Example A: The Dreaded Algebra Word Problem

Think about how hard it is for students to set up an equation that can be used to solve an algebra word problem. Some reasons for the difficulties include reading levels and unfamiliar contexts. But there has to be more to it than these surface features.

Consider, for example, the following two problems.

Example A: The Dreaded Algebra Word Problem

1. The driving distance from Boston to Chicago is is 990 miles. Sendhil drives from Boston to Chicago at an average speed of 50mph and returns at an average speed of 60mph. For how many hours is Sendhil on the road?

2. Sendhil drives from Boston to Chicago at an average speed of 50mph and returns at an average speed of 60mph. Sendhil is on the road for 36 hours. What is the driving distance from Boston to Chicago?

The problems have identical reading levels, and the context is the same in each. But teachers report that many students who can solve problem 1 are baffled by problem 2.

Example A: The Dreaded Word Problem

This is where the reasoning habit of “expressing the rhythm” in a calculation can be of great use. The basic idea:

- Guess at an answer to problem 2, and
- check your guess as if you were working on problem 1, *keeping track of your steps*.

The purpose of the guess is not to stumble on (or to approximate) the correct answer; rather, it is to help you construct a “checking algorithm” that will work for any guess.

Example A: The Dreaded Word Problem

Problem 2: Sendhil drives over at an average speed of 50mph and returns at an average of 60mph. He's on the road for 36 hours. What is the driving distance?

So, you take several guesses until you are able to express your checking algorithm in algebraic symbols. For example:

- Suppose the distance is 1000 miles.
- How do I check the guess of 1000 miles? I divide 1000 by 50. Then I divide 1000 by 60. Then I add my answers together, to see if I get 36. I don't.
- So, check another number—say 950. 950 divided by 50 plus 950 divided by 60. Is that 36?
- No, but a general method is evolving that will allow me to check *any* guess.

Example A: The Dreaded Word Problem

- My guess-checker is

$$\frac{\text{guess}}{50} + \frac{\text{guess}}{60} \stackrel{?}{=} 36$$

- So my *equation* is

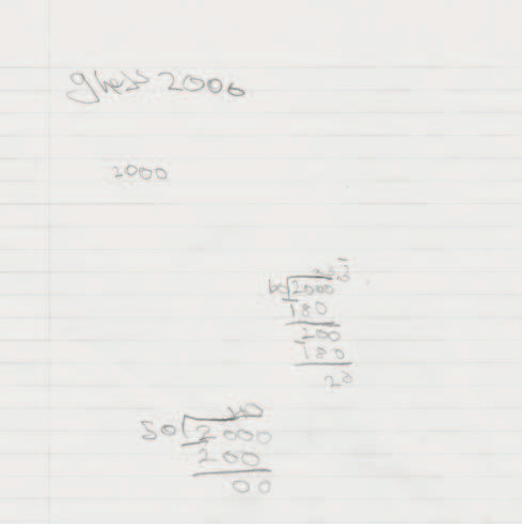
$$\frac{\text{guess}}{50} + \frac{\text{guess}}{60} = 36$$

or, letting x stand for the unknown correct guess,

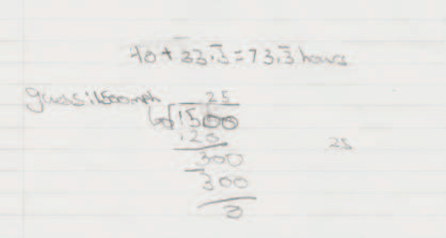
$$\frac{x}{50} + \frac{x}{60} = 36$$

EXAMPLE A: THE DREADED WORD PROBLEM

Here's some student work that shows how the process develops:



Example A: The Dreaded Word Problem

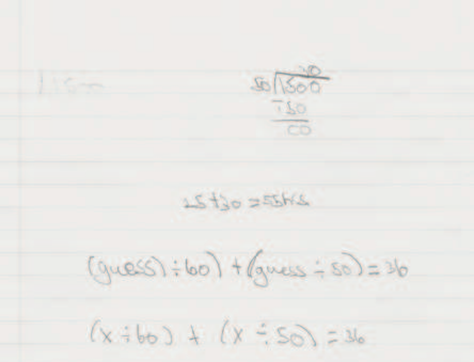


$40 + 33.\bar{3} = 73.\bar{3}$ hours

guess: 1500 mph

$$\begin{array}{r} 25 \\ 60\overline{)1500} \\ 120 \\ \hline 300 \\ 300 \\ \hline 0 \end{array}$$

25



$$\begin{array}{r} 30 \\ 50\overline{)1500} \\ 150 \\ \hline 00 \end{array}$$

$25 + 30 = 55$ hrs

$(\text{guess}) \div 60) + (\text{guess} \div 50) = 36$

$(x \div 60) + (x \div 50) = 36$

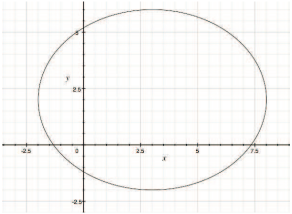
Example B: Equations for lines

The phenomenon was first noticed in precalculus . . .
Graph

$$16\,x^2 - 96\,x + 25\,y^2 - 100\,y - 156 = 0$$

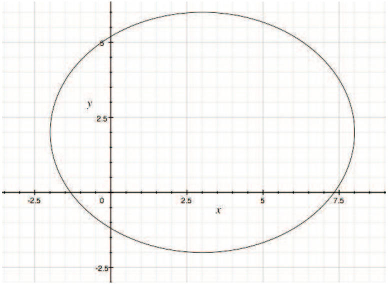
$$16\,x^2 - 96\,x + 25\,y^2 - 100\,y - 156 = 0 \Rightarrow \frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$$

$$\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1 \Rightarrow$$



Example B: Equations for lines

$$\frac{(x-3)^2}{25}+\frac{(y-2)^2}{16}=1$$



Is $(7.5, 3.75)$ on the graph?

This led to the idea that "equations are point testers."

Example B: Equations for lines

- Suppose a student, new to algebra and with no formulas in tow, is asked to find the equation of the vertical line ℓ that passes through $(5, 4)$.
- Students can draw the line, and, just as in the word problem example, they can guess at some points and check to see if they are on ℓ.
- Trying some points like $(5, 1)$, $(3, 4)$, $(2, 2)$, and $(5, 17)$ leads to a generic guess-checker:

  To see if a point is on ℓ, you check that its x-coordinate is 5.

- This leads to a guess-checker: $x \stackrel{?}{=} 5$ and the equation

$$x = 5$$

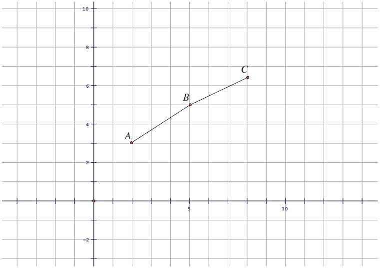
Example B: Equations for lines

- What about lines for which there is no simple guess-checker? The idea is to find a geometric characterization of such a line and then to develop a guess-checker based on that characterization. One such characterization uses *slope*.
- In first-year algebra, students study slope, and one fact about slope that often comes up is that three points on the coordinate plane, not all on the same vertical line, are collinear if and only if the slope between any two of them is the same.

Example B: Equations for lines

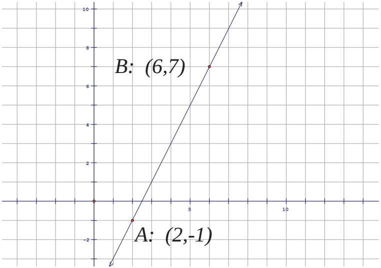
If we let $m(A, B)$ denote the slope between A and B (calculated as change in y-height divided by change in x-run), then the collinearity condition can be stated like this:

Basic assumption: A, B, and C are collinear $\Leftrightarrow m(A, B) = m(B, C)$



Example B: Equations for lines

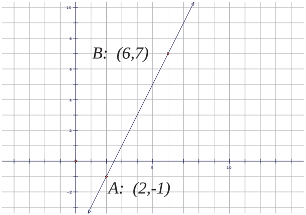
What is an equation for $\ell = \overleftrightarrow{AB}$ if $A = (2, -1)$ and $B = (6, 7)$?



Try some points, keeping track of the steps...

Example 2: Equations for lines

- $A = (2, -1)$ and $B = (6, 7)$
- $m(A, B) = 2$



- Test $C = (3, 4)$:
  $m(C, B) = \frac{4-7}{3-6} \stackrel{?}{=} 2 \Rightarrow$ Nope
- Test $D = (5, 5)$:
  $m(D, B) = \frac{5-7}{5-6} \stackrel{?}{=} 2 \Rightarrow$ Yup
- The “guess-checker?”
  Test $P = (x, y)$:
  $m(P, B) = \frac{y-7}{x-6} \stackrel{?}{=} 2$

And an equation is $\frac{y-7}{x-6} = 2$

Other Examples Where This Habit Is Useful

- Finding lines of best fit
- Building expressions (“*three less than a number*”)
- Fitting functions to tables of data
- Deriving the quadratic formula
- Establishing identities in Pascal’s triangle
- Using recursive definitions in a CAS or spreadsheet

$\vdots$

The Complete Correlation

- Correlation with all eight practice standards will soon be released

Common Core State Standards for High School	Meeting the Common Core State Standards with Pearson's *CME Project*
Standards for Mathematical Practices	
1. Make sense of problems and persevere in solving them.	The "experience before formality" principle in *CME Project* has sense making as one of its main goals. We see to it that definitions and theorems are capstones, not foundations, and that students spend plenty of time playing with ideas in simple and transparent cases before the ideas are formalized and made precise. The student dialogues play a role here, too, giving examples of how students gradually refine insights through perseverance and through attempts to communicate with others. Algebraic expressions are purposefully transformed so that students can see where an expression is zero or where it is optimized. Stamina is developed in several ways—recurring themes throughout the program and open-ended, long-term projects at the end of every chapter. **Algebra 1:** For examples, see 1.1, 1.11, 2.6, Ch. 2 Project, 5.1, 5.7, 5.10, 6.1, 7.5, 7.11, Ch. 7 Project, 8.5, 8.10 **Geometry:** For examples, see 1.0, 1.6 (Ex. 15), 1.11, 2.4, 2.9, 2.14, 5.4, 8.2, 8.5, 8.6 **Algebra 2:** For examples, see 1.1, 1.3, 1.5, 1.6, 1.8, 1.9, 1.12, 2.2, 2.4, 2.6, 2.7, 2.10, 2.13, 2.14, 3.3, 3.11, 4.2, 4.6, 5.3, 6.3, 6.7, 6.9, 7.2, 7.3, 8.11, 8.13 **Precalculus:** For examples, see 1.5, Ch. 1 Project, 2.5, 4.1, 4.5, 4.8, Ch. 4 Project, 5.11, Ch. 5 Project, 6.13, 7.4, 8.2, 8.4

- A draft is on the handout

CME and the Content Standards

We really don't have much to do.

In Algebra 1, for example, we need to tweak some of the lessons to add

- the recommended method for solving linear inequalities
- correlation vs. causation and two-way frequency tables
- a bit about the sum of a rational and irrational number

It's the same level of change in the other courses.

Conjecture: This paucity of new content is a consequence of the coherence that comes from a focus on mathematical habits of mind.

The Complete Correlation

- Correlation with the content standards will soon be released

Common Core State Standards for High School	Meeting the Common Core State Standards with Pearson's *CME Project*
Use properties of rational and irrational numbers.	
N-RN.3 Explain why sums and products of rational numbers are rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.	**Algebra 1 Lesson coverage** 1.13, 1.15, 6.10 **Algebra 1 Standards Activity** In Lesson 6.10, have students answer the following questions. 1. Show that the sum of two rational numbers is rational. [If $\frac{a}{b}$ and $\frac{c}{d}$ are rational, so that a, b, c, and d are integers and b and d are not zero, then by the basic rules, $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$. Both the numerator and denominator are integers, and $bd \neq 0$ since neither $b \neq 0$ nor $d \neq 0$. Therefore $\frac{ad + bc}{bd}$ is rational.] 2. Show that the product of two rational numbers is rational. [If $\frac{a}{b}$ and $\frac{c}{d}$ are rational, so that a, b, c, and d are integers and b and d are not zero, then by the basic rules, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. The numerator and denominator are integers, and $bd \neq 0$ since neither $b \neq 0$ nor $d \neq 0$. Therefore $\frac{ac}{bd}$ is rational.] 3. Show that $\frac{1}{2} + \sqrt{5}$ is irrational. [If $\frac{1}{2} + \sqrt{5}$ were a rational number, say $\frac{p}{q}$ then $\sqrt{5} = \frac{p}{q} - \frac{1}{2}$ by the basic rules. But the right-hand side is rational by Exercise 1, and the left-hand side is irrational. Since this cannot be true, $\frac{1}{2} + \sqrt{5}$ is irrational.]

CME Principles with Respect to CCSS

- Stay faithful to the organizing principles
  - Focus on mathematical practice
  - Place high school mathematics in the broader landscape
- Keep the focus on *mathematical* coherence.
  - Algebra courses should still emphasize algebraic themes and practices, . . .
  - Tools and methods should still have “legs”
- Listen carefully
  - to the amazing teachers who are using the program
  - to our advisors
  - to our publisher

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