# TOWARDS A CURRICULUM DESIGN BASED ON MATHEMATICAL THINKING

AL CUOCO

## Introduction

Curriculum design in US precollege mathematics is largely topic driven; a course is defined by the topics it treats. And the major criteria for including a topic in any particular course include:

- does it review and deepen important ideas from previous courses?
- is it a prerequisite for likely subsequent courses?
- did it fall through the cracks in earlier grades?
- is it on the state test?

As one moves up the grades, the effects of this design principle compound. By the time one reaches high school, we end up with 18-chapter compendia of topics that range from graphing equations to triangle trigonometry to data analysis to complex numbers. These monster texts have become *de facto* definitions for the American high school curriculum.

Of course, there's much more in each of these texts than what students will ever need for future courses and what teachers can possibly teach in a year. Indeed, it's a well-known fact among high school teachers that one can only finish slightly more than half of these chapters in a given year, and yet many students who go on to the next course from such experiences have what they need to get respectable grades.

But in addition to being too big, these courses are, at a deeper level, too small. There has been a growing consensus among all involved in secondary mathematics education that this topic-driven curriculum is not serving our students well:

1. The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more importantly, from developing the ways of thinking—*the habits of mind*—used by scientists, mathematicians, engineers, and in other professions in which mathematics is a core ingredient. Explicit emphasis on essential mathematical habits such as reasoning by continuity, abstracting regularity from repeated calculations, developing theories based on numerical evidence, and using thought-experiments are all but missing from many American programs.
2. Mathematics as a scientific discipline is one of the crowning achievements of the human intellect. The RAND study panel [5] put it this way:

   > Mathematics constitutes one of the most ancient and noble intellectual traditions of humanity. It is an enabling discipline for all of science and technology, providing powerful tools for analytical thought as well as the concepts and language for precise quantitative descriptions of the world. Even the most elementary mathematics involves knowledge and reasoning of extraordinary subtlety and beauty.

Much of this writing is taken from other papers, including [1] and [2].

> But after decades of sustained and creative efforts, there is still a wide disconnect between school mathematics and mathematics as a scientific discipline. At the recent joint AMS-MAA meetings in Atlanta, talk after talk gave examples of how precollege courses emphasize low-level details with the same importance as essential mathematical results and how there is no overall mathematical point to many of the problems that students work. In far too many classrooms, from elementary school through undergraduate school, mathematics is taught as a disconnected set of facts and procedures, a body of knowledge to be learned in much the same way as one learns a list of terms for a vocabulary test.

Of course, this analysis is not new to the developers of IMD curricula. NSF has invested heavily in development efforts aimed at helping students become mathematical thinkers and at showcasing the immense utility and beauty of our subject.

The purpose of this paper is to describe the newest such effort, one that is similar in spirit and goals to the existing NSF-funded programs but that is based on a somewhat different philosophy and that uses a different set of organizing principles. Building on two prior curriculum efforts [12, 13], my colleagues and I are developing a four-year high school program, the *CME Project*.

## Some design features of the *CME Project*

**Traditional course structure.** The *CME Project* is a student-centered and problem-based program that adheres to the traditional American course structure: its courses have titles like Algebra, Geometry, and Precalculus. In addition to the logical reasons (described below) for such an organization, there is considerable evidence, both from discussions with the teachers with whom we work and from several studies ([14, 15], for example) that, while high school teachers want to use problem-based materials and new methods of teaching, they are not as motivated to switch to integrated curricula or to unfamiliar organizations. So, the first reason for our approach is that there seems to be some demand in the field for it.

But another reason is that the ways in which ideas are organized in mathematics itself—essentially around the themes of algebra, geometry, and analysis—have emerged over the centuries as a scheme for bringing coherence to the discipline. Many standard curricula look at each of these areas as sets of results and techniques. Many integrated programs look at them as organizers that run through varying contexts. The *CME Project* team sees these branches of mathematics not only as compartments for certain kinds of results, but also as descriptors for *methods* and *approaches*—the habits of mind that determine how knowledge is organized and generated within mathematics itself. As such, they deserve to be centerpieces of a curriculum, not its byproducts [3].

**The role of applications.** "Power users" of mathematics are able to see abstract connections among seemingly different phenomena and can synthesize mathematical methods, often in unorthodox ways. And the first step in developing this proficiency is to expand students' conceptions of the "real world" to *include* mathematics.

This principle was crystallized for us several years ago when we invited some high school seniors to a meeting to discuss an earlier version of what will become our course for seniors [12]. The meeting took place at the end of term 1; up to that point, the students had been fitting polynomial functions to tables by looking at successive differences and other interpolation methods, and they had been proving that a recursively defined function and its closed form were equal on positive integers, using mathematical induction [11]. We asked them to tell us what was most different about this course. About four students (out of 12) said, almost in unison, some variation of "it's more realistic." Others agreed. This reaction surprised even us, and as we poked at the remarks a bit, we discovered that the students meant that they were doing realistic *work*. These students were typically not the "best" in the school—most of them did not have the grades to get into high school calculus—but they were motivated not by a context but by the chance to use their own mathematical thinking.

Students in the *CME Project* apply elementary algebra and mathematical induction to determine the monthly payment on a loan; they use complex numbers as a device for establishing trigonometric identities; they use elementary arithmetic to study methods for creating secure ciphers; they apply Euclidean geometry to perspective drawing, optimization problems, and trigonometry. *All* these situations are applications of mathematics, because the emphasis is on *how* one uses mathematics as opposed to *where* one uses it.

**The fundamental dialectic: Open and closed.** The *CME Project* is the direct descendent of two previously developed courses, each using the traditional course structure and each focussing on mathematical thinking.

*Connected Geometry* [13] was developed immediately after the release of the 1989 *Standards* [6], and it reflected many of the attitudes about curriculum that arose in that liberating period of American education. The book contained few stated theorems and definitions, and even fewer worked-out examples. The theorems, definitions, and examples were all there, but they were in the teacher's edition or the solution guide. What students saw was a collection of activities and provocative problems that were designed to help them discover the results for themselves.

Teachers who were not part of the original field tests told us that *Connected Geometry* was too much of a guide and not enough of a reference. They loved the open-ended problems, but they felt that the activities needed more closure—*in the student text.*

*Mathematical Methods* [12] was developed around the time that NCTM was revising the *Standards* in preparation for *PSSM* [7]. It, too, was a product of its time, influenced by the growing sense that all students needed a robust technical fluency (with algebraic and numerical calculations) and that students should be able to refer to their text as a resource for results and examples. *Mathematical Methods* contained many more proved theorems, worked-out examples, and "practice" exercises than we included in *Connected Geometry.*

With the *CME Project*, we have developed a design that is both informed by our previous work and faithful to both needs: students can use their texts as both a guide and a reference. We have come to realize that, at the high school level, understanding develops in two important ways: as the result of independent (or guided) investigations and as the result of reading, discussing, and internalizing mathematical exposition. Each *CME Project* activity starts out with a problem set that students do *before* instruction and that provides experiments that preview—in simple numerical and geometric contexts—the important ideas in the exposition. The lesson then includes worked-out examples or written dialogues that codify methods, bring closure to this experimentation, and provide a reference for later work. In addition, each lesson has a set of *orchestrated practice problems* in which students practice arithmetic and algebraic skills while they try to abstract off some regularity that suggests an interesting mathematical result.

**Developing mathematical habits.** Habits of mind are just that—habits that take time to develop. The *CME Project*'s organization provides students the time and focus they need to develop central mathematical ways of thinking. But can beginning high school students really "think like mathematicians?"

We are convinced that they can. While the problems and methods used by research mathematicians are out of reach for most non-specialists, one of the wonderful things about our discipline is that the modes of thought used on the frontiers of what's known are natural extensions of ordinary human thought. And it turns out that developing this "mathematical mindedness" can go a long way to help students overcome what seem to be stubborn misconceptions about mathematics.

For example, the difficulties that students have with algebra word problems are legendary. A *mathematical* analysis of the difficulties students have in this area shows that the obstacles are related to the mathematical habit that's in play when one abstracts off regularity from repeated calculations and compiles the actions into a coherent process, defining a mathematical function [4]. In our own high school teaching, we have exploited this common mathematical habit to develop a rather effective method (currently called "guess, check, and keep track of your steps") that helps students model situations with algebraic expressions.

Another example: the "Cartesian connection" between geometry and algebra is so ingrained in mathematical thinking that its often a surprise to see how underdeveloped it is in algebra students, even among juniors and seniors. But we have seen repeatedly in high school classes that many high school students do not understand the fact that one can test a point to see if it is on the graph of an equation by seeing if its coordinates satisfy the equation. Just *knowing* that this is a problem helps the *CME Project* writers design specific (and recurring) types of problems that address it. In the *CME Project*, we have developed a method (currently called the "point-tester method") that helps students derive equations for geometric objects by finding algebraic characterizations for the coordinates of the points on the objects.

For more on the habits of mind perspective, see [8]. For more examples of curricular implications, see [1, 2, 9, 10, 16]

**High expectations.** Reviews of early drafts of the *CME Project* chapters invariably contain comments like "high school kids could never do this kind of thing." Field tests show otherwise. The materials have to be revised, and in many cases completely reworked, but in no case do we need to water down the level of mathematics for either students or teachers. We are convinced that traditional curricula expect far too little from teachers and students. and that students at all levels can do this kind of work. Much of the field test of *Mathematical Methods*, *Connected Geometry*, and now the *CME Project* Algebra 1 has taken place in "ordinary" classes; I co-taught two sections of *Mathematical Methods* for two years to students who were the weakest students in their school taking a fourth year of mathematics, and they were able to rise to my expectations. Poor performance in mathematics courses has many causes, but lack of ability to think in a characteristically mathematical way is, for the vast majority of students, not one of them. The *CME Project* design employs a *low threshold–high ceiling* approach: each chapter starts with activities that are accessible to all students and ends with problems that will challenge the most advanced students. It's often (pleasantly) surprising to see how far students take the materials.

**The role of technical fluency.** The *CME Project* team takes the position, coming from our teaching experience and work as mathematicians, that, for the vast majority of students, the development of technical expertise in numerical and algebraic calculation is corequisite with the development of conceptual mathematical understanding. Of course, every teacher knows that the development of technical expertise does not guarantee mathematical understanding—we have all seen examples of students who can calculate like the wind and yet who cannot apply their skills to solve problems or develop theories.

But a great deal of algebraic thinking involves reasoning about calculations in algebraic structures, and in order to reason about calculations, students need to have some experience in performing them.

And reasoning about calculations in abstract symbol systems is *useful*. For example, a manual for a commercial spreadsheet contains hundreds of examples, presented in a notation quite removed from traditional algebraic notation, that require this kind of algebraic thinking to understand. Finding one's way around the Internet, working through income tax software, even making flexible use of a word processor are all made easier by a knack for reasoning about operations and developing theories of calculations in self-contained systems.

In the *CME Project*, we invite students to become fluent in algebraic calculations so that they can reason about them. For example, we ask students to investigate the distribution of possible sums when 3 (or more) dice are thrown. We then ask them to find the coefficient of $x^9$ when

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^3$$

is expanded. The object here is *not* to perform the expansion by hand or machine, but to calculate without calculating, reasoning where an $x^9$ term can occur when one multiplies out the expression.

When we have tried this with teachers and students, invariably someone says something like, "It's the number of ways you can make 9 as a sum of three numbers between 1 and 6—it's the number of ways you can roll a nine when 3 dice are thrown!" Such reasoning requires a kind of "decontextualization"—a formal approach to polynomial calculations that is important enough to deserve increased attention in the later years of high school.

**Involving the community.** From the time we wrote *Connected Geometry*, we have taken the perspective that criticism from every corner of the mathematical community is essential to our work. We invite teachers, mathematics educators, and mathematicians to be on advisory boards, to consult with us, and to review the materials, and we take care to include people who are likely to have different points of view.

The *CME Project* has a teacher advisory board that meets monthly during development cycles. It includes teachers who say they will never use a text that has worked-out examples and teachers who say that students should never tackle a problem unless they are given instruction on how to solve it. Of course we can't follow either of these extremes, but this tension has had a substantial influence on our design, leading to the current structure of preliminary problem sets that are assigned with no instruction, the use of dialogues to convey exposition, the "for you to try" element that follows every worked-out example, and a host of other design elements.

Our national advisory board includes people with very diverse perspectives: Hy Bass, Dick Askey, Eric Robinson, Barbabra Janson, Hung Hsi Wu, Herb Wilf, Glenn Stevens, Art Heinricher, Ed Barbeau, Arthur Eisenkraft, and Jackie Miller, among others. Roger Howe is an advisor as well as a core consultant. In addition to making for *very* spirited advisory board meetings, the varying perspectives we get from these advisors, consultants, and reviewers has greatly enriched out work and has made it highly likely that we hear about errors, criticisms, and points of view different from ours *before* the materials hit the street. As with our monthly teacher meetings, we don't take all the advice we get, but even suggestions that are not implemented have an indirect effect on the finished product. In this way, we believe that the program reflects common wisdom across the entire community.

IMD developers know how valuable field testing is to the process. The *CME Project* is especially fortunate to work with a group of teachers who provide us with detailed feedback and criticism and who guide the development in ways that can only come from people who work every day with the materials. We are very grateful for the hard work of

- Jaime Lao and Calvin Baylon (Alexander Graham Bell Jr. High, CA)
- Karena Hanscom, Dale Erz, John Henry, Kathy Greer (Port Angeles High School, WA)
- Jayne Abbas (David R. Cawley Middle School, NH)
- Rocky Cundiff (Ignacio High School, CO)
- Maureen Mulryan (Lowell High School, MA)
- Annette Roskam (Rice Lake High School, WI)

Another important segment of the mathematics community is the publishing industry. Here, too, we think it is essential to incorporate the best advice we can get. Early in the development, we began talking to publishers with the express intention of establishing a genuine partnership that would complement our expertise. The *CME Project* will be published by Prentice Hall, and we have benefitted greatly from our collaboration with Stewart Wood, Kathy Carter, and Elizabeth Lehnertz. These colleagues bring more than a knowledge of design, marketing, and field testing—they also have a real sense for how schools work and a refined taste for good mathematics.

## Conclusion

Like any curriculum, the *CME Project* is not for everyone. The structure of the courses, emphasis on mathematical habits of mind, and approach to how mathematics is applied will appeal to some teachers and not others. The development of specific mathematical topics (trigonometry, say), will resonate with some and grate on the nerves of others. Our taste in topics and focus is certainly not the only one. But the research we did prior to applying to NSF for funding convinces us that there *is* a sizable audience for a program like the *CME Project*. And, if the program appeals to these teachers, it's likely that the mathematics in it will be understood by their students.

One final remark: The quality of a curriculum depends, more than on any design feature or consultant or underlying philosophy, on the quality of the staff. The *CME Project* team includes includes amazingly talented people: writers, teachers, mathematicians, and educators. It's a wonderful experience to work with

- Nancy D'Amato,
- Daniel Erman,
- Brian Harvey,
- Wayne Harvey,
- Jud Hill,
- Jane Gorman,
- Karen Graham,
- Bowen Kerins,
- Helen Lebowitz,
- Melanie Palma, and
- Kevin Waterman

## References

[1] "Preparing for Calculus and Beyond: Some Curriculum Design Issues". To be published by the Mathematical Association of America in the MAA Notes volume, *Rethinking the Road toward Calculus*.

[2] "Beyond Representation" in Greenes, Carole and Carol Findell, Eds. (2005) *Developing students' algebraic reasoning abilities*. Volume 3 in the NCSM-Houghton Mifflin Company School Division and McDougal Littell Monograph Series for Leaders in Mathematics Education. Boston, MA: Houghton Mifflin Company.

[3] Schmidt, W., Houang, R., and Cogan, L. (2002). "A Coherent Curriculum." The American Educator, Summer, 2002.

[4] Action to Process: Using Functions to Solve Algebra Word Problems. *Intelligent Tutoring Media*, 4:4, 1993.

[5] Ball, D., et.al. (2002). *Mathematical Proficiency for All Students: Toward a Strategic Research and Development Program in Mathematics Education.* http://www.rand.org/multi/achievementforall/math/:OERI

[6] National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for Teaching Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

[7] National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics.* Reston, VA: Author.

[8] Cuoco, A., Goldenberg, E.P., and Mark, J. (1996). Habits of Mind: An Organizing Principle for Mathematics Curriculum. *Journal of Mathematical Behavior*, 15(4).

[9] Cuoco, A. (1998). Mathematics as a Way of Looking at Things. In *High School Mathematics at Work*, Washington, DC: National Academy Press.

[10] Cuoco, A. and Levasseur, K. ( ). Classical Mathematics in the Age of CAS. In *Computer Algebra Systems in School Mathematics*, NCTM (2004).

[11] Cuoco, A. and Manes, M. (2001). When Memory Fails: Putting Limitations to Good Use. *The Mathematics Teacher*, 94(6). Preprint available online at `<http://www.edc.org/CME/>`.

[12] Education Development Center, Inc. (2001). *Mathematical Methods: Topics in Discrete and Precalculus Mathematics*. Armonk, NY: It's About Time, Inc. Development supported by NSF Grant ESI 9617369.

[13] Education Development Center, Inc. (2000). *Connected Geometry.* Chicago: Everyday Learning Corporation.

[14] St. John, M., Fuller, K.A., Houghton, N., Huntwork, D., and Tambe, P. (2000) *High School Mathematics Curricular Decision-Making: A National Study of How Schools and Districts Select and Implement New Curricula.* http://www.inverness-research.org/reports/ab_compassmonog.html: Inverness Research Associates.

[15] St. John, M., et al. (2000) COMPASS: *Curricular Options in Mathematics Programs for All Secondary Students.* Preliminary evaluation report. Inverness Research Associates.

[16] Dynamic Geometry as a Bridge from Euclidean Geometry to Analysis. In *Geometry Turned On*, J. King and D. Schnattseider (eds), MAA Notes 41. 1997.

Center for Mathematics Education, EDC, Newton, MA 02458, `alcuoco@edc.org`