Applications of CAS (and other) Technology to Topics in High School Mathematics

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Slides available at `http://www.edc.org/CME/showcase/`

Examples Taken From "The CME Project"

a comprehensive four-year NSF-funded high school curriculum forthcoming from Prentice Hall

with additional support from Texas Instruments

The CME Project Approach

(1) "Traditional" course structure: it's familiar but different

- each course focusses on particular mathematical habits
- each course uses examples, contexts from many fields
- each course is organized around mathematical themes

(2) The audience: the (large number of) teachers who

- want a problem- and exploration-based program
- want to bring activities to "closure"
- want the familiar course structure

Some Features of the CME Project

(1) *It is faithful to mathematics as a discipline.*

(2) *It highlights the uses of mathematics.*

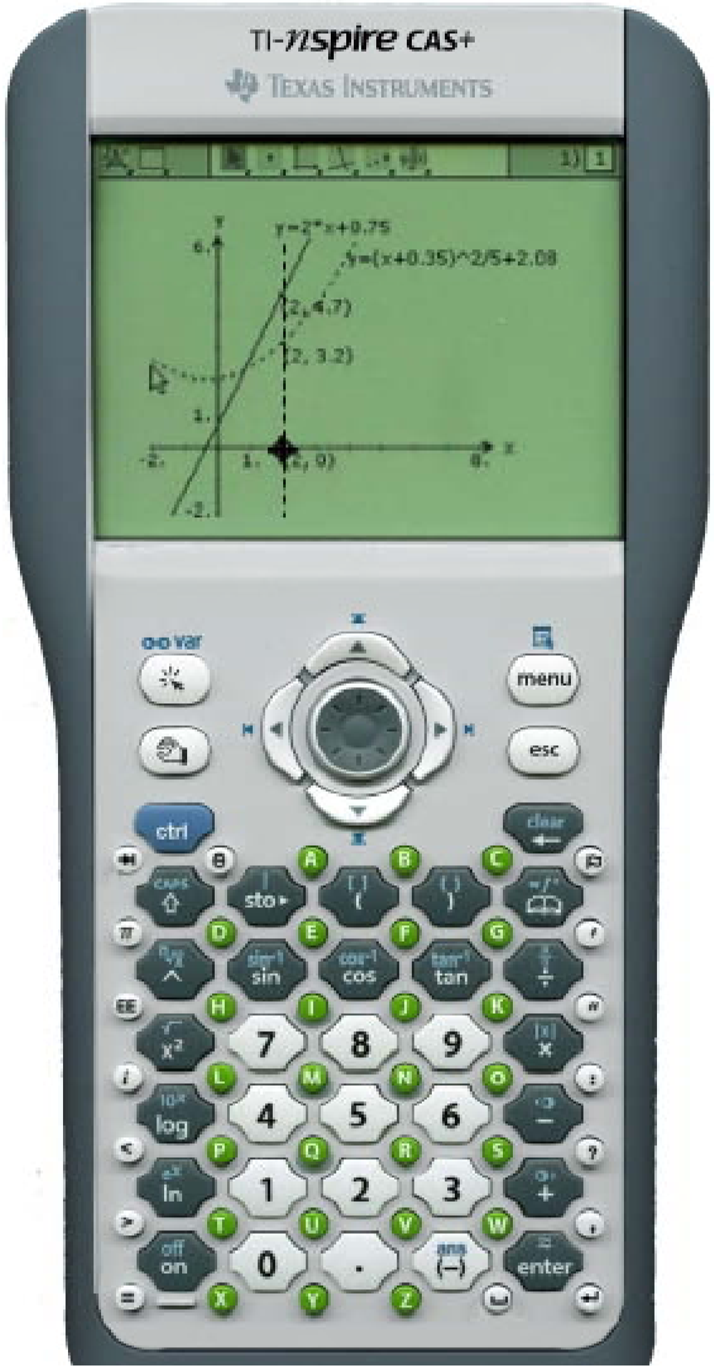
(3) *It is organized around mathematical habits of mind.*

(4) *Students use the text as both a guide and as a reference.*

(5) *It makes essential use of technology:*

- A "function-modeling" language
- A Computer Algebra System
- A dynamic geometry environment

These exist in the TI-***n****spire*. Today, we'll look at the first two.


TI-nspire CAS+
Texas Instruments
y=2*x+0.75
y=(x+0.35)^2/5+2.08
(2, 4.7)
(2, 3.2)
menu
esc
ctrl
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sin
cos
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log
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on
enter
7
8
9
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5
6
1
2
3
0

Why CAS-based technology?

Such systems

- make tractable and enhance many beautiful classical topics, historically considered too technical for high school students.
- support experiments with algebraic expressions and other mathematical objects in the same way that calculators can be used to experiment with numbers.
- allow students to build computational models of algebraic objects and structures that have no faithful physical counterparts.

The CMP Factor Game

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30

The CME Project Factor Game

$x - 1$	$x^2 - 1$	$x^3 - 1$	$x^4 - 1$	$x^5 - 1$
$x^6 - 1$	$x^7 - 1$	$x^8 - 1$	$x^9 - 1$	$x^{10} - 1$
$x^{11} - 1$	$x^{12} - 1$	$x^{13} - 1$	$x^{14} - 1$	$x^{15} - 1$
$x^{16} - 1$	$x^{17} - 1$	$x^{18} - 1$	$x^{19} - 1$	$x^{20} - 1$
$x^{21} - 1$	$x^{22} - 1$	$x^{23} - 1$	$x^{24} - 1$	$x^{25} - 1$
$x^{26} - 1$	$x^{27} - 1$	$x^{28} - 1$	$x^{29} - 1$	$x^{30} - 1$

If n is a non-negative integer, how *many* irreducible factors over the integers does $x^n - 1$ have?

In other words, we're looking for a pattern in the outputs of the function

$$n \mapsto \text{ \# of factors of } x^n - 1 \text{ over } \mathbb{Z}$$

n	Number of Factors of $x^n - 1$
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

Finding patterns takes some work. Explaining them takes one into fairly deep waters. The point here is that the CAS allows one to perform algebraic experiments in much the same way that a scientific calculator can be used to perform numerical experiments.

Theorem: If n is a positive integer, we have a formal identity:

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$$

This is sometimes called the *cyclotomic identity* and is sometimes written

$$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + x + 1$$

If n is prime, the second factor is irreducible over $\mathbb{Z}$ (Eisenstein).

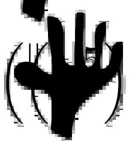
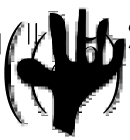
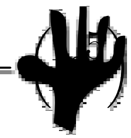
If n is composite, the second factor splits some more—

Its factorization is quite wonderful and mysterious.

The cyclotomic identity helps explain the algebra factor game. For example,

$$x^{15} - 1 = \left(x^5\right)^3 - 1$$

$= ($  $)^3 - 1$

$= (($  $) - 1)(($  $)^2 + ($  $) + 1)$

$$= \left(\left(x^5\right) - 1\right)\left(\left(x^5\right)^2 + \left(x^5\right) + 1\right)$$

So

$$x^{15} - 1 = \left(x^5 - 1\right)\left(x^{10} + x^5 + 1\right)$$

The CME Project Factor Game, Version 2

$x-1$	$x+1$	x^2-1	x^2+x+1	x^3-1
x^2+1	x^4-1	$x^4+x^3+x^2+x+1$	x^5-1	x^2-x+1
x^6-1	$x^6+x^5+x^4+x^3+$ $x^2+x+2+x+1$	x^7-1	x^4+1	x^8-1
x^6+x^3+1	x^9-1	$1-x+x^2-x^3+x^4$	$x^{10}-1$	$x^{10}+x^9+\dots$ $+x+1$
$x^{11}-1$	x^4-x^2+1	$x^{12}-1$	$x^{12}+\cdots+x+1$	$x^{13}-1$
$1-x+x^2-x^3+$ $x^4-x^5+x^6$	$x^{14}-1$	$1-x+x^3-x^4+$ $x^5-x^7+x^8$	$x^{15}-1$	$x^{16}-1$

Monthly Payments

Suppose you want to buy a car that costs \$10,000. You don't have much money, but you can put \$1000 down and pay \$350 per month. The interest rate is 5%, and the dealer wants the loan paid off in two years. What kind of car can you buy?

This leads to the question

"How does a bank figure out the monthly payment on a loan?"

or

"How does a bank figure out the balance you owe at the end of the month?"

Take 1 (not very realistic)

What you owe at the end of the month is what you owed at the start of the month minus your monthly payment.

So, if

$b(n, m) =$ the balance at the end of month n with a monthly payment of $\$m$

then

$$b(n, m) = \begin{cases} 9000, & n = 0 \\ b(n-1, m) - m, & n > 0 \end{cases}$$

The model on the TI- ***n****spire* looks exactly the same ...

Take 2

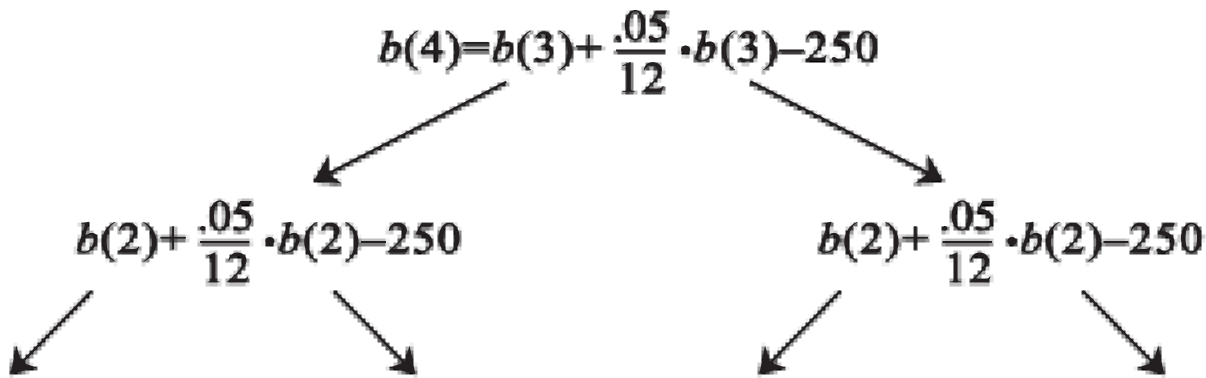
What you owe at the end of the month is what you owed at the start of the month, *plus* $\frac{1}{12}$ *of the yearly interest on that amount*, minus your monthly payment.

$$b(n,m) = \begin{cases} 9000, & n = 0 \\ b(n-1,m) + \frac{.05}{12} b(n-1,m) - m, & n > 0 \end{cases}$$

Students can then use successive approximation to make

$$b(24, ???) = 0$$

Except . . .



It takes too much %!*&$ work

Take 3

Same as Take 2, with a little algebraic simplification for efficiency:

What you owe at the end of the month is what you owed at the start of the month, *plus* $\frac{1}{12}$ *of the yearly interest on that amount*, minus your monthly payment.

$$b(n,m) = \begin{cases} 9000, & n = 0 \\ \left(1 + \frac{.05}{12}\right) b(n-1,m) - m, & n > 0 \end{cases}$$

Students then use successive approximation to make $b(24, ???) = 0$.

Let's try it:

What's the monthly payment

on a loan of $9000 for 24 months?

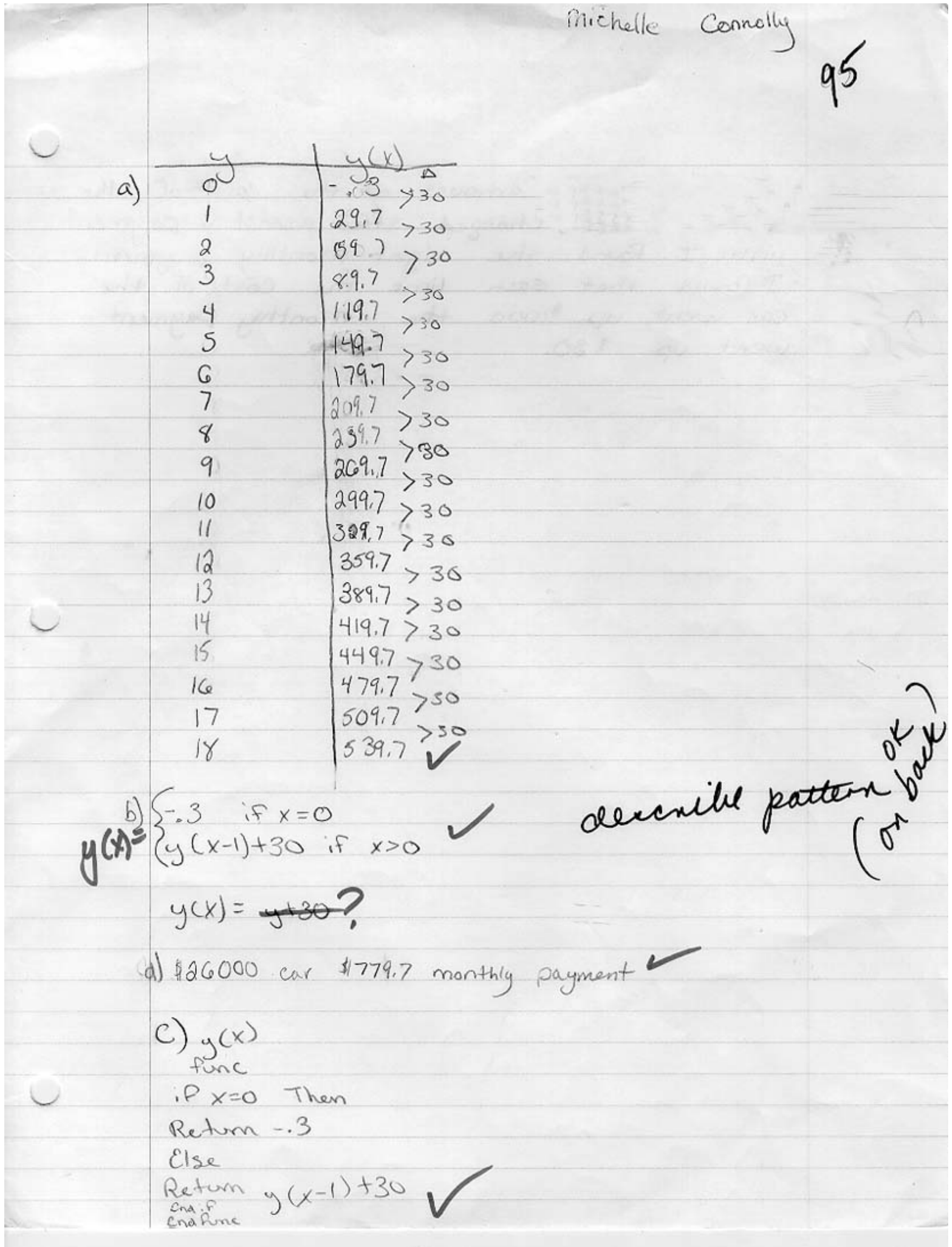
Pick an interest rate and keep it constant. Suppose you want to pay off a car in 24 months. How does the monthly payment change with the cost of the car?

(1) Make a table like this:

Cost of car (in thousands of dollars)	Monthly payment
10	
11	
12	
13	
14	
⋮	⋮

(2) Describe a pattern in the table. Use this pattern to find either a closed form or a recursive rule that lets you calculate the monthly payment in terms of the cost of the car in thousands of dollars. Model your function with your CAS and use the model to to find the monthly payment on a \$26000 car. Check your result with the original "`b`" program

- I changed the amount of the cost of the car then I changed the monthly payment until I found the right monthly payment.
- I found that each time the cost of the car went up $1000 the monthly payment went up $30.



Michelle Connolly

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a)

y	y(x)	Δ
0	-.3	> 30
1	29.7	> 30
2	59.7	> 30
3	89.7	> 30
4	119.7	> 30
5	149.7	> 30
6	179.7	> 30
7	209.7	> 30
8	239.7	> 30
9	269.7	> 30
10	299.7	> 30
11	329.7	> 30
12	359.7	> 30
13	389.7	> 30
14	419.7	> 30
15	449.7	> 30
16	479.7	> 30
17	509.7	> 30
18	539.7	✓

b) $y(x) = \begin{cases} -.3 & \text{if } x=0 \\ y(x-1)+30 & \text{if } x>0 \end{cases}$ ✓

describe pattern (OK on back)

y(x) = ~~y+30~~ ?

d) \$26000 car \$1779.7 monthly payment ✓

c) y(x)

func

if x=0 Then

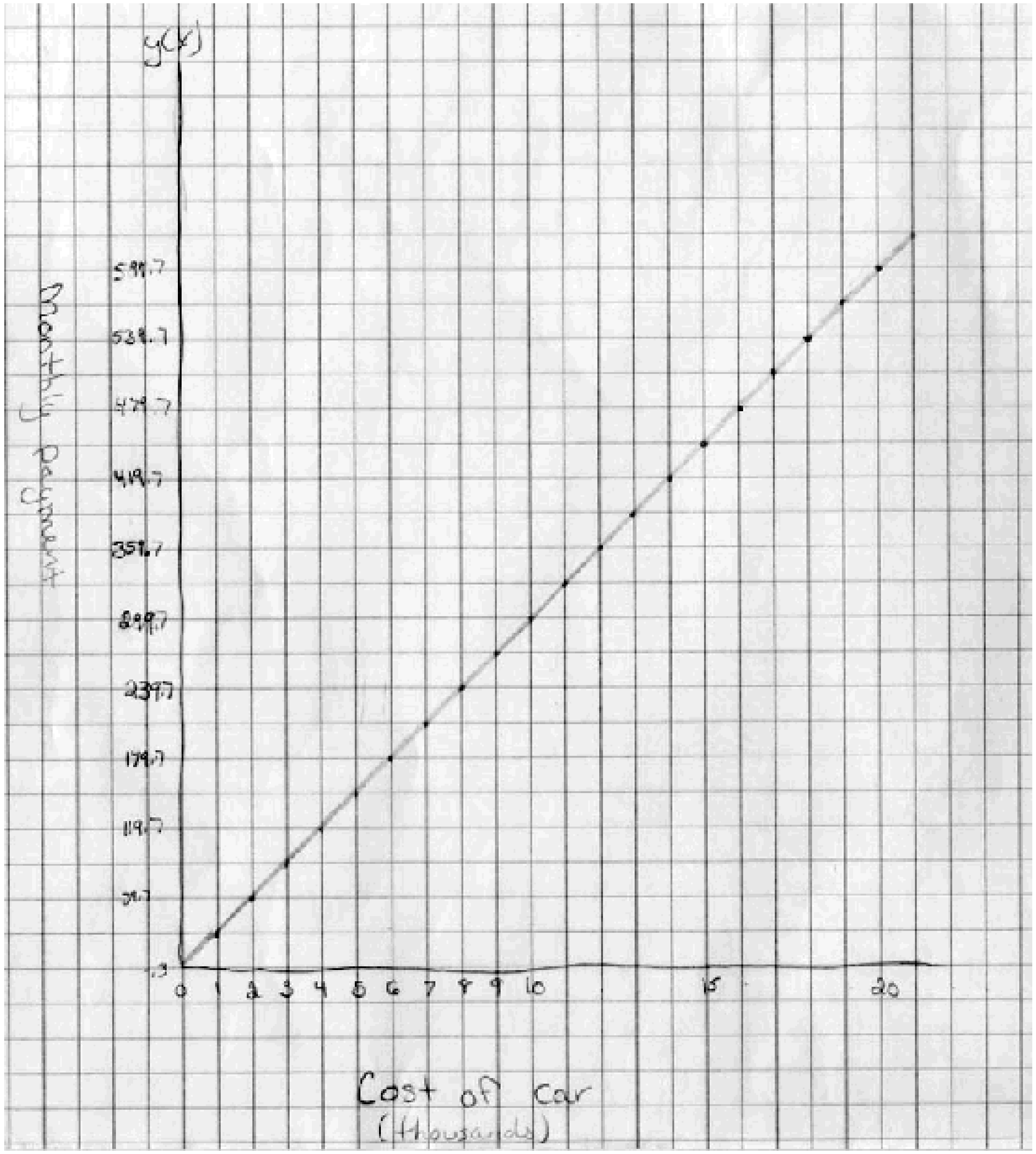
Return -.3

Else

Return y(x-1)+30 ✓

end if

end func


y(x)
Monthly Payment
599.7
539.7
479.7
419.7
359.7
299.7
239.7
179.7
119.7
59.7
.3
0 1 2 3 4 5 6 7 8 9 10
15
20
Cost of car
(thousands)

The balance at the end of 24 months with a monthly payment of \$350 can be gotten by entering `b(24,350)` in the calculator:

```
:                b(24,350)

:               1129.40
```

$\vdots$

Because we are using a CAS, we can do it *generically*: The balance at the end of 24 months with a monthly payment of m can be gotten by entering `b(24,m)` in the calculator:

```
:                   b(24,m)

:                  9944.47202003 -25.185920534*m
```

So,

```
:                 solve(9944.47202003 -25.185920534*m,m)
:                 m = 394.842...
```

A monthly payment of \$394.84 will do the trick.

Suppose you borrow \$9000 at 5% interest. Then you are experimenting with this function:

$$b(n,m) = \begin{cases} 9000 & \text{if } n = 0 \\ (1 + \frac{.05}{12}) \cdot b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

Notice that

$$1 + \frac{.05}{12} = \frac{12.05}{12}$$

Call this number q. Then

$$b(n,m) = \begin{cases} 9000 & \text{if } n = 0 \\ q \cdot b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

where q is a constant. And you want a value of m so that $b(24,m) = 0$.

You could "unstack" the calculation as follows:

$$
\begin{aligned}
b(24,m) &= q \cdot b(23,m) - m \\
&= q\,(q \cdot b(22,m) - m) - m = q^2 \cdot b(22,m) - qm - m \\
&= q^2\,(q \cdot b(21,m) - m) - qm - m \\
&\qquad = q^3 \cdot b(21,m) - q^2 m - qm - m \\
&\quad \vdots \\
&= q^{24} \cdot b(0,m) - q^{23}m - q^{22}m - \cdots - q^2 m - qm - m \\
&= 9000 \cdot q^{24} - m(q^{23} + q^{22} + \cdots + q^2 + q + 1)
\end{aligned}
$$

But

$$q^{23} + q^{22} + \cdots + q^2 + q + 1 = \frac{q^{24} - 1}{q - 1} \quad \text{(The Cyclotomic ID!)}$$

Sooo . . .

$$b(24, m) = 9000\, q^{24} - m\frac{q^{24} - 1}{q - 1}$$

Setting $b(24, m)$ equal to 0 gives an explicit relationship between m and the cost of the car:

$$m = 9000\, \frac{(q-1)q^{24}}{q^{24} - 1}$$

In general,

$$\text{montly payment} = \text{cost of car} \times \frac{(q-1)q^{n}}{q^{n} - 1}$$

where n is the term of the loan and

$$q = 1 + \frac{\text{interest rate}}{12}$$

And we can model this on our CAS . . .

Other uses of CAS and n-spire function language in the CME Project:

- Mathematical induction
- Fitting functions to data
  - Lagrange interpolation
  - Newton's difference formula
- Sequences and series
  - sums of powers
  - summing differences
  - sequences of *polynomials* (multiple angle formulas)
- Difference equations
  - recursive models
  - closed-form solutions